



**OPEN EDUCATIONAL
RESOURCES
4
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Taking Education to the People



Open Educational Resources (OER) for Open Schooling

The Commonwealth of Learning (COL) Open Schools Initiative launched an Open Educational Resources (OER) Project to provide materials under the Creative Commons license agreement to support independent study in 17 specially selected secondary school subjects. Funded by the William and Flora Hewlett Foundation its aim is to broaden access to secondary education through the development of high quality Open Distance Learning (ODL) or self-study materials.

These specially selected OER subjects include:

1. Commerce 11
2. Coordinated Science 10 (Biology, Chemistry and Physics)
3. English 12
4. English Second Language 10
5. Entrepreneurship 10
6. Food & Nutrition
7. Geography 10
8. Geography 12
9. Human Social Biology 12
10. Life Science 10
11. Life Skills
12. Mathematics 11
13. Mathematics 12
14. Physical Science 10
15. Physical Science 12
16. Principles of Business
17. Spanish

Open Educational Resources are free to use and increase accessibility to education. These materials are accessible for use in six countries: Botswana, India, Lesotho, Namibia, Seychelles and Trinidad & Tobago. Other interested parties are invited to use the materials, but some contextual adaptation might be needed to maximise their benefits in different countries.

The *OER for Open Schooling Teachers' Guide* has been developed to guide teachers/instructors on how to use the Open Educational Resources (OER) in five of these courses.

1. English
2. Entrepreneurship
3. Geography
4. Life Science
5. Physical Science

The aim of this teachers' guide is to help all teachers/instructors make best use of the OER materials. This guide is generic, but focuses on Namibian examples.

Print-based versions are available on CD-ROM and can be downloaded from www.col.org/CourseMaterials. The CD-ROM contains the module and folders with additional resources, multimedia resources and/or teacher resources. Note that not all subjects have multimedia resources.

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Published by:

Commonwealth of Learning
1055 West Hastings, Suite 1200
Vancouver, British Columbia
Canada V6E 2E9
Telephone: +1 604 775 8200
Fax: +1 604 775 8210
Web: www.col.org
Email: info@col.org

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Mathematics

Grade 11

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**Zambia College of Distance Education
Mathematics Department**

Private bag 20
Luanshya
Copperbelt
Zambia

Fax: +260 212 510271

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Newton Ng'ambi	Course developer
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About This self-study workbook

Mathematics Grade 11 has been produced by Zambia College of Distance Education. All self-study workbooks produced by Zambia College of Distance Education are structured in the same way, as outlined below.

How This self-study workbook is Structured

The Course Overview

The course overview gives you a general introduction to the course. Information contained in the course overview will help you determine:

- If the course is suitable for you.
- What you will already need to know.
- What you can expect from the course.
- How much time you will need to invest to complete the course.

The overview also provides guidance on:

- Study skills.
- Where to get help.
- Course assignments and assessments.
- Activity icons.
- Units.

We strongly recommend that you read the overview *carefully* before starting your study.

The Course Content

The course is broken down into units. Each unit comprises:

- An introduction to the unit content.
- Unit outcomes.
- New terminology.
- Core content of the unit with a variety of learning activities.
- A unit summary.
- Assignments and/or assessments, as applicable.

Resources

For those interested in learning more on this subject, we provide you with a list of additional resources at the end of this self-study workbook; these may be books, articles or websites.

Your Comments

After completing this Mathematics Course we would appreciate it if you would take a few moments to give us your feedback on any aspect of this course. Your feedback might include comments on:

- Course content and structure.
- Course reading materials and resources.
- Course assignments.
- Course assessments.
- Course duration.
- Course support (assigned tutors, technical help, etc.)

Your constructive feedback will help us to improve and enhance this course.

Course Overview

Welcome to Mathematics Grade

11

Welcome to the Mathematics 11 course. We hope you will enjoy this course and that by the end of this course your mathematical knowledge and skills will be well developed. Congratulations on your decision to commit yourself, over the next months, to seriously studying this course.

This course covers fifteen (15) units. You should study all the units as you will be examined for your accreditation at the end of grade 12 on these units and the units you studied in grade 10.

The **first unit** in this course is on Approximations and Scientific notation. This unit consists of three topics. In this unit you will learn how to round off measurements to the nearest stated units and write numbers correct to the required number of decimal places and significant figures. You will also learn how to write numbers in scientific notation (standard form) and further solve problems on estimation of errors.

Unit 2 is on relations and functions. The unit has two topics. In these topics you will learn how to relate members of two or more sets according to a given rule. You will also learn to how to express relations using function notation and find the inverse of the function.

In Unit 3 you will learn about the graphs of polynomials. This unit consists of three topics. You will learn how to draw and interpret the graphs of linear, quadratic and cubic functions. You will also learn how to find the maximum and minimum values of the quadratic and cubic functions. You will further learn how to calculate the area under the curve and the gradient of the curve at a given point.

Unit 4 is on quadratic equations. The unit has two topics in which you will learn how to solve quadratic equations by using the factorisation method, completing the square method, quadratic formula and the graphical method.

In Unit 5 you will learn about ratio, proportion and rate. The unit consists of four topics. In this unit you will learn how to express two different numbers or quantities as ratios in their simplest form. You will also learn how to solve problems involving direct and inverse proportion. You will further learn how to solve environmental problems involving ratio, proportion and rate.

Unit 6 is on variation. The unit consists of two topics. In this unit you will learn how to solve problems on direct and inverse variation. You will also learn how to draw and interpret graphs of direct and inverse variation. You will further learn how to solve problems based on joint and partial variation.

In Unit 7 you will learn about travel graphs. The unit has two topics. You will first learn how to draw and interpret distance-time graphs and

speed-time graphs. You will then learn how to calculate distance, time, speed, acceleration and retardation from travel graphs.

Unit 8 is on symmetry in two dimensions. The unit has two topics in which you will learn how to identify line of symmetry and rotational symmetry in two dimensional shapes. Thereafter you will learn how to determine the order of rotational symmetry in two dimensional shapes. You will further identify symmetry inbuilt in the natural environmental.

Unit 9 is on congruence and similarity. The unit is divided into three topics. In this unit you will learn how to identify congruence and similar triangles. You will learn how to calculate the unknown sides or angles in similar figures. You will further learn how to calculate area and volume of similar figures.

Unit 10 is on angle properties of a circle and polygons. This unit has two units. In this unit you will learn how to calculate the angles using angle properties of a circle. You will also learn how to calculate angles using properties of triangles and quadrilaterals. You will further learn how to calculate the sizes and the sum of exterior and interior angles of polygons.

Unit 11 is on Mensuration. The unit is divided into three topics. First, you will learn how to calculate the area and perimeter of plane figures. Then you will learn how to calculate the surface area and volume of three dimensional figures. Lastly, you will learn how to draw nets of cones, cubes, cuboids, pyramids and cylinders.

In **Unit 12** you will learn about locus of points in two and three dimensions. The unit consists of two topics. You will learn how to determine the locus of a point in two dimensions. You will learn how to construct to locus of points from a given point and from two given points. You will further learn how to solve problems on locus of points in three dimensions.

Unit 13 is on trigonometry. The unit is divided into two topics. In this unit you will learn how to find the three trigonometric ratios of sine, cosine and tangent of acute angle. You will learn how to solve problems involving right-angled triangles using sine, cosine and tangent ratios. You will further learn how to apply trigonometric ratios to calculate angles, height and distances and solve problems involving three figure bearing.

Unit 14 is on statistics. The unit has two topics. In these topics you will learn how to collect and classify data and present it in form of tables, pie charts, bar graph, line graph and histogram. You will also learn how to find mean, median and mode of a set of data.

Unit 15 is on sequences. This is the last unit in your Grade 11 Mathematics course. The unit has two topics. In this unit you will learn how to recognise and continue a sequence. You will also learn how to identify patterns within and across different sequences. You will further learn how to generalise sequences in simple algebraic statements to the n th term.

After completing all fifteen units of this course, we hope you will be able to use the knowledge and skills gained in your daily life. We also hope

that this course will whet your appetite for further study of Mathematics in Grade 12 and in tertiary or higher education.

Mathematics Grade 11: Is this course for you?

This course is intended for people who:

1. Want to improve their academic qualifications in order to gain entry into tertiary education.
2. Want to gain knowledge and skills in solving mathematical and related problems
3. Want to continue their secondary education through the distance education mode because they want to study while working or engaged in businesses or cannot afford to go into full time studies due to financial constraint or other reasons.

To study this course you need to have:

1. Studied mathematics at junior secondary school level and obtained Junior Secondary School Leaving Certificate and
2. Completed Grade 10 Mathematics course successfully.

Course Outcomes



Outcomes

Upon completion of the Mathematics Grade 11 course you will be able to:

- *Demonstrate* spatial awareness in two and three dimensions.
- *Explore* mathematical situations.
- *Apply* mathematical knowledge to problem solving.
- *Estimate, measure and calculate* to an appropriate degree of accuracy using suitable methods.
- *Recognise* patterns and structures in a variety of situational forms and justify generalisations.
- *Represent, interpret and use* data in a variety of situations.

Timeframe



How long?

How long each unit takes will depend a lot on each individual learner. We estimate that this course can be completed in 30 weeks or 210 hours. This gives you, on average, approximately 13 hours to study each unit, including completing the in-text activities and 3 hours per tutor-marked assignments. The number of hours per unit is however flexible. You might spend less time on shorter units and more time on the longer units.

Besides the 13 hours that you will spend studying this unit, you will require time for attending tutorials. The college will introduce you to the study centre near your home. You are encouraged to visit the study centre at least once every two weeks to attend tutorials and join study groups. The duration of tutorials is about 2 hours per tutorial. You should, however, get the details of the tutorial programme from the tutors at the study centre.

Study Skills



As an adult learner your approach to learning will be different to that from your school days. You will choose what you want to study, you will have professional and/or personal motivation for doing so. You will most likely be fitting your study activities around other professional or domestic responsibilities as well.

Essentially you will be taking control of your learning environment. As a consequence, you will need to consider performance issues related to time management, goal setting, stress management, etc. Perhaps you will also need to reacquaint yourself in areas such as essay planning, coping with exams and using the web as a learning resource.

Your most significant considerations will be *time* and *space* i.e. the time you dedicate to your learning and the environment in which you engage in that learning.

We recommend that you take time now—before starting your self-study—to familiarize yourself with these issues. There are a number of excellent resources on the web. A few suggested links are:

- <http://www.how-to-study.com/>

The “How to study” web site is dedicated to study skills resources. You will find links to study preparation (a list of nine essentials for a good study place), taking notes, strategies for reading text books, using reference sources, test anxiety.

- <http://www.ucc.vt.edu/stdysk/stdyhlp.html>

This is the web site of the Virginia Tech, Division of Student Affairs. You will find links to time scheduling (including a “where does time go?” link), a study skill checklist, basic concentration techniques, control of the study environment, note taking, how to read essays for analysis, memory skills (“remembering”).

- <http://www.howtostudy.org/resources.php>

Another “How to study” web site with useful links to time management, efficient reading, questioning/listening/observing skills, getting the most out of doing (“hands-on” learning), memory building, tips for staying motivated, developing a learning plan.

The above links are our suggestions to start you on your way. At the time of writing these web links were active. If you want to look for more go to www.google.com and type “self-study basics”, “self-study tips”, “self-study skills” or similar.

Need help?



Help

Zambia College of Distance Education has recently transformed itself into an open and distance learning institution and that some of the resources and services are still being developed. Some of these resources and services are a learner’s resource centre, Internet connectivity and institutional establishment for the position of personnel like librarian/ research assistant, course instructor, teaching assistant and an IT specialist (one whom you can contact for technical issues such as computer problems, website access, etc).

In case you need help, you can get in touch with the college either by visiting or phoning the college at +260 212 51027 during working hours (between 08:00 and 17:00hrs). The college is situated at plot No. 41, Independence Avenue, Luanshya. The mathematics tutors will be ready to attend to you on problems that relate to your course.

For inquiries, you can get in touch with the secretary to the principal or registry staff, using the above stated contact details.

The college (ZACODE) has arrangements with local schools and resource centres for you to use their resources. The college has also arranged that annually learners will be given separately information on the centres that are available near their homes.

Assignments



Assignments

All the topics in this unit have an End of Topic Exercise which is located in the assignment section. The number of topic exercises in each unit ranges from 2 to 4. In total there will be 37 topic exercises in this course. You are expected to complete all these topic exercises.

Most of the topic exercises could last between 2 hours and 3 hours. You are encouraged to complete the exercise within the estimated time. The time you spend on the exercises correspond to the time you will spend in the final examinations. A mathematics paper in the examinations lasts for 2 hours 30 minutes. Before you begin answering the questions you should take note of the time you start answering the questions and the time you complete answering the questions.

After completing the exercises you must mark your own work by comparing your answers with model answers provided at the end of the unit.

Assessments



Assessments

There are six (6) Tutor Marked Assessments for this course. There will be one assessment attached to Unit 3, Unit 6, Unit 9, Unit 12 and Unit 15. These can be found in the Assessment section of this course.

The assessments must be submitted to the college by using the postal services or the study centres. At the study centres, these assessments can be submitted to the tutors or the college staff.

The time to submit the assessments is open. You have to submit the assessments as soon as the work in the prescribed units has been completed.

The order of the assessments to submit is determined by the units studied. The order of submission is as follows:




















- The first assessment is for Units 1, 2 and 3;
- The second assessment is for Units 4, 5 and 6
- The third assessment is for Units 7, 8 and 9,
- The fourth assessment is for Units 10, 11 and 12,
- The fifth assessment is for Units 13, 14, and 15.
- The sixth assessment is an end of year assessment, covering the total course.

Getting Around This self-study workbook

Margin Icons

While working through this self-study workbook you will notice the frequent use of margin icons. These icons serve to “signpost” a particular piece of text, a new task or change in activity; they have been included to help you to find your way around this self-study workbook.

A complete icon set is shown below. We suggest that you familiarize yourself with the icons and their meaning before starting your study.

			
Activity	Assessment	Assignment	Case study
			
Discussion	Group activity	Help	Note it!
			
Outcomes	Reading	Reflection	Study skills
			
Summary	Terminology	Time	Tip
			
Objectives	Feedback	Answers to Assessments	

Unit 1

Approximations and Scientific Notation

Introduction

Welcome to unit 1 in your mathematics 11 course. This unit is the first of fifteen units you are going to study in grade 11. These units are very important in your course because you will be tested at the end of your course together with other units you learned in Grade 10 and those you will learn in grade 12 in order for you to qualify for the general certificate of education.

This unit has three topics. The content of these topics are not totally new to you since they were covered in your Junior Certificate Program. In each topic you will be required to do some activities and feedback will be provided immediately thereafter. At the end of each topic there is a topic exercise. You are encouraged to answer all the questions in the topic exercises and to then check your work against the model answers only after completing the exercises.

Unit 1 does not have an end of unit assignment. You will only be required to do the three end of topic exercises for the three topic of this unit. It is only at the end of unit 3 that you will be required to complete a tutor marked assessment 1 based on the first three units and which you will send to your tutor for marking.

Upon completion of this unit you will be able to:

- *Approximate* quantities.
- *Approximate* measures to a given degree of accuracy.
- *Write* numbers correct to the required number of decimal places.
- *Write* numbers correct to the required number of significant figures.
- *Write* numbers in scientific notation (Standard form).
- *Calculate* estimates of error.
- *Use* approximations to estimate an error.



Outcomes

Timeframe



How long?

We estimate that to complete this unit you will need between 11 to 15 hours. This time includes the time you will spend in doing the activities and checking them against the feedback. If you do not finish studying the unit within this estimated time do not worry since we do not all learn at the same pace.

You are encouraged to spend about 2 hours answering each topic exercise in this unit. Since there are three topic exercises in this unit, this means that you will spend about 6 hours on these exercises.

The total hours for completing the unit will thus be between 17 and 22 hours.

Learning Resources



Tip

In order to study this unit with minimal difficulties you will need the following materials:

- A ruler graduated in centimetres
- A pencil or a pen
- A calculator

Teaching and Learning Approaches



Tip

In this unit we have used three teaching and learning methods in presenting the content. These methods are:

- **Conceptual:** This method will help you understand the meaning of facts, rules, formulas and procedures;
- **Problem-solving:** This method will help you solve mathematical problems that relate to real life situation. You will also be able to discuss mathematical problems, answers and strategies with friends.
- **Skills:** This method will help you practice using the facts, rules, formulas and procedures by doing self-marked activities and a topic exercise.

As you read through the unit, do the activities and/or exercises and discuss your ideas with other learners and your tutor. By doing this you will be putting into practice the above three teaching and learning

methods. When you consistently apply these methods, we trust that you will achieve greater understanding of the unit and be able to relate the knowledge to your real life situation.



Terminology

Round off:	To express a number as the nearest significant number above or below it for ease of calculation.
Unit:	A defined amount of a quantity, serving as a basis for measuring other amounts of the same quantity, e.g. the metre is defined as a unit of length.
Decimal number:	A number that has decimal point which divides the whole numbers from the tenths, hundredths, and smaller divisions of ten.
Recurring decimal:	The decimal point which does not end.
Significant figure:	A digit in a numeral that is relevant to the value of the number represented.
Absolute error:	The difference between the true measure and the one obtained by measurement.
Relative error:	The ratio of the absolute error in a measurement to the size of the measurement.
Tolerance:	The difference between the greatest and the least acceptable measurements.

Topic 1 Approximations

This topic is, as mentioned above, the first of three topics in this unit. In this topic you will learn how to round off numbers to a given degree of accuracy. Approximation is one of many interesting topics in this mathematics course in the sense that most of the units in your Mathematics course use approximations.

Approximations are not new to you as you learned about them in the Junior Certificate program. This time you will learn this topic in detail. You will learn how to round off measures to the nearest unit and correct numbers to the required number of decimal places. You will further learn how to express numbers to the required number of significant figures. After studying each topic you will be required to do a self assessment exercise at the end of the topic. You are encouraged not to go through the feedback before doing the topic exercises.

In this first topic of the unit, we will address the first four of the seven unit outcomes.

Upon completion of this topic you will be able to:



Outcomes

- *Approximate* quantities.
- *Approximate* measures to a given degree of accuracy.
- *Write* numbers correct to the required number of decimal places.
- *Write* numbers correct to the required number of significant figures.

Approximate Quantities

Approximation of quantities is one of the things we always do in our daily life. We approximate the numbers of things we have or see, the mass of things in kilograms, and the amount of time in hours or in minutes we spend doing an activity and so on. For instance, we estimated that if you study this unit and do all the activities, you will need approximately 13 hours. You may not spend exactly 13 hours studying this unit. You may spend either less or more time than the estimated time depending on your speed.

Whenever we measure a physical quantity like time, length, volume, temperature or mass there is always some degree of error or uncertainty. No matter how careful we are in carrying out a measurement, we can never know what the correct measure is. This is the reason why approximations are important in mathematics. To illustrate this fact let us do the following activity. In this activity you will be required to read the time as precisely as you can.

Let us do the first activity that involves rounding off time.



Activity 1

The following diagram is a clock showing the time in the afternoon. State the time and write down the time in the space below.

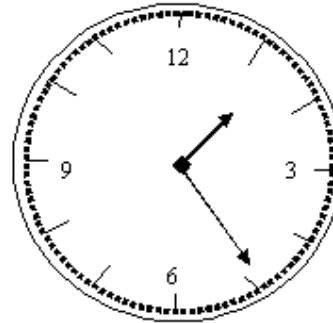


Figure 1: Diagram showing time in the afternoon

Write your answer in the space given below.



Feedback

Feedback

You might not have found it easier to read the time as exactly as you can. You might, like us, have read the time to approximate 13:24 hours correct to the nearest minutes.

Whenever we are asked to measure the given length we may not all find the same measurements due to some errors. To illustrate this let us do another activity. This time you will need some friends to do an activity. Two friends will do.



Group activity

You will need two friends and a ruler to do the following activity. In this activity you will need to measure the height of a triangle.

The activity we will do this time is about measuring the height of the triangle. You will need a ruler to do this activity.



Activity 2

Use a ruler to measure the height of the triangle ABC given below. Three of you should measure the height of the triangle in turns.

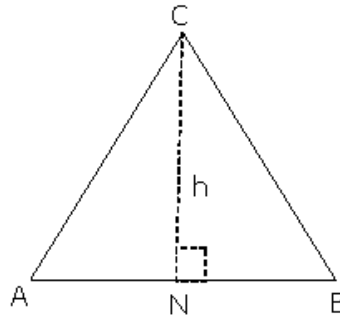


Figure 2: Diagram showing a triangle ABC

Write your answers in the space below.

Your measurement: _____

First friend's measurement: _____

Second friend's measurement: _____

Now compare your measurements. Have you noticed anything on your measurements?



Feedback

Feedback

You may have discovered that your measurements are not the same. They might differ slightly.

From this illustration we can learn that whenever we take measurement, there is always some degree of error. No matter how careful we are in carrying out a measurement, we can never know what the correct measure is. This is the reason why we round off numbers to the nearest units.

When we measure the physical quantity we are only able to take the approximate measurement due to the limitations of our measuring instruments, the conditions under which the measurement is made and the ways the operator uses the instrument. This is why approximations are important in Mathematics.

There are several ways in which approximations can be made. These include rounding off the measures to:

- The nearest unit or fraction of a unit.
- A number of decimal places.

- The required number of significant figures.

We will take you through these ways one at a time. Let us begin with rounding off the measurements to the nearest unit or fraction of a unit.

Round Off Numbers to The Nearest Unit

We said earlier in this topic that whenever we take measurements, we always round off measures to the nearest unit. To round off numbers we are guided by the rule which states that **if the number preceding the figure to be rounded off is greater than or equal to 5, then add 1 to the previous digit.**

Now you will be taken through some activities in which you will round off measurements to the nearest unit. We will begin by rounding off temperature to the nearest degree. Measuring temperature is commonly done in our daily lives especially when we visit the health centre or clinics for medical treatment.



Activity 3

The diagram below shows part of a mercury in-glass thermometer. Study the diagram and write down the temperature in the space below.

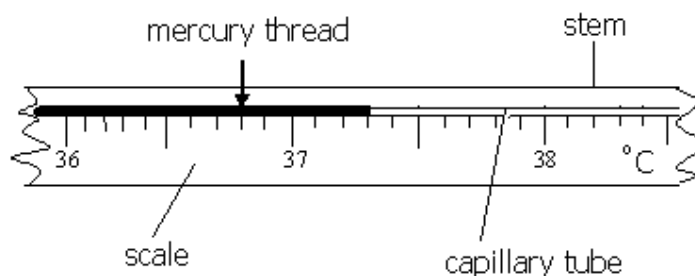


Figure 3: Part of a mercury in-glass thermometer

Once you have read out the temperature you can write down your answer in the space below.



Feedback

Feedback

Like us, you may have written that the temperature shown in the diagram is 37.3°C.

Let us now see how we arrived at this answer. The temperature is between 37 and 38. The fraction part of this temperature is 3.

Since the number 3 (last digit) to be rounded off is less than 5 then we will round down the temperature.

Therefore, 37.3°C = 37°C correct to the nearest degree

Note that we have rounded off this temperature to the nearest 1 degree.

Let us do another activity. This activity is similar to the previous one. It is about rounding off mass to the nearest kilogram. Finding the mass of the objects is not new to you since we do it almost every day especially when buying food stuff in the supermarket.



Activity 4

Consider the mass 334.9 kg. Write down this mass correct to the nearest kilogram in the space provided below.



Feedback

Feedback

In your answer to the activity you would have used the rule of rounding off to round off this number. When we examined the number carefully we discovered that the figure to be rounded off is 9 which is greater than 5, then we had to add 1 to the previous digit.

Thus, 334.9 kg = 335 kg, to the nearest kilogram

Let us do another activity. This activity is similar to the previous one. The only difference is that this time the measurement is in centimetres.



Activity 5

Round off the length 14.73 cm to the nearest millimetre. Write down your answer in this space.



Feedback

Feedback

If you have examined this measurement carefully you might have noticed that the length given is in centimetres and we are required to round off the length to the nearest millimetre.

To solve this problem we have to change centimetres to millimetres first and then we can proceed with our workings as follows:

Since, $10 \text{ mm} = 1 \text{ cm}$

Therefore, we can convert 14.73 cm to millimetres by multiplying it by 10 as follows:

Thus, $14.73 \times 10 = 147.3 \text{ mm}$

Now we can round off 147.3 mm to the nearest millimetre as follows:

Therefore, $147.3 \text{ mm} = 147 \text{ mm}$ correct to the nearest millimetre.

The number 3 to be rounded off is less than 5

The length 147 mm can be expressed in centimetres as before as 14.7cm by dividing it by 10.

Since $10 \text{ mm} = 1 \text{ cm}$,

Then, $147 \text{ mm} \div 10 = 14.7 \text{ cm}$

Therefore, $147.3 \text{ mm} = 14.7 \text{ cm}$

*Remember the **decimal part** of the measurement represents **millimetres**.*

Let us do another activity together. This is similar to the previous one. This time the measurement has two digits after decimal point.



Activity 6

Consider the volume 182.34 litres and round it to the nearest tenth of a litre. Write down your answer and working in this space.



Feedback

Feedback

You should have noted that the volume 182.34 litres is expressed as a decimal fraction. The decimal part of this fraction is 3 tenths and 4 hundredths of a litre.

*You should have noted as well that the 4 hundredths to be rounded off is less than 5. Therefore, the 4 hundredths at the end of the number is rounded down and the volume 182.34 litres will be expressed as **182.3** litres to the nearest tenth of a litre.*

Let us do another activity together. This is similar to the previous one. This time the measure has three digits after decimal point.



Activity 7

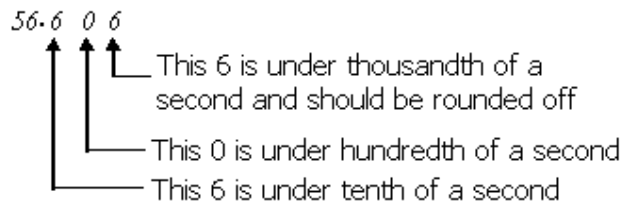
Consider the time 56.606 seconds carefully and then round it off to the nearest hundredth of a second. Write down your answer in the space below



Feedback

Feedback

In this activity we should first determine the place value of the fraction part of 56.606 seconds. The word “place value” is not new to you since you learned about it in your Junior Certificate Program. The place value can be worked out as follows:



*When we examine the last digit in the number 56.606 seconds carefully, we will discover that the 6 thousandth in the decimal part of the number is greater than 5. You should remember the rule that you learned in the first part of this topic which states that: **If the figure to be rounded off is 5 or greater than 5, then we add 1 to the previous digit.***

Therefore, the number 56.606 will be rounded up by adding 1 to 0 hundredth to give 1 hundredth.

*Thus, 56.606 = **56.61**seconds, to the nearest hundredth of a second*

We have so far learned to round off numbers to the nearest unit and the fraction of the unit. Now let us discuss how to write numbers correct to the required number of decimal places.

Round Off Decimal Fractions to Stated Decimal Places

The numbers we have just looked at in Activities 3 – 7 are referred to as decimal fractions. You should note that a decimal fraction has a decimal point which divides the whole numbers from fraction (the tenths, hundredths, and smaller divisions of ten). The place or specific number of digits to the right of the decimal point in a line of numbers is referred to as **decimal place**.

Let us look at the numbers we have just learned and determine their decimal places. These numbers are:

- 147.3 mm has one decimal place.

- 182.34 litres has two decimal places.
- 56.606 seconds has three decimal places.

The decimal fractions can be written to a certain degree of accuracy by rounding off decimal places to a given number. We can do this by reducing the number of decimal places. Now, let us look at how to round off decimal fractions to the required number of decimal places.

- The first number, 147.3 mm has already been rounded off to the least number of decimal place; that is one decimal place.
- The second number, 182.34 litres can be rounded off to one decimal place. When we round off to one decimal place the answer becomes **182.3 litres**.

Note that the last digit, 4, has been rounded off. Since it is less than 5 then we do **not** add 1 to the previous digit.

- The third number, 56.606 seconds can be rounded off either to one decimal place or to two decimal places.

Thus, 56.606 seconds = **56.61 seconds**, correct to two decimal places.

Note that the last digit, 6, that we have to round off is greater than 5. We will therefore add 1 to the previous digit, 0, to get 1.

Or

56.606 seconds = **56.6 seconds**, correct to one decimal place.

Note that we have to round off the last two digits, 0 and 6. Since zero to be rounded off is less than 5, then we will not add 1 to the previous one.

Now you can do the following activity to apply and consolidate what you have just learned.



Activity 8

Round off the number 25.044 to 2 decimal places. Write down your answer and working in the space provided below.



Feedback

Feedback

You would have noticed that the number 25.044 has 3 decimal places. By rounding off this number to 2 decimal places we drop off the last digit. Since the number we are dropping off is less than 5 then the previous digit will remain as it is.

Therefore, $25.044 = 25.04$, to 2 decimal places

You should do another activity. This activity is similar to the previous one.



Activity 9

Round off the number 69.999 to 2 decimal places. Write down your answer and working in this space.



Feedback

Feedback

When we examine the number 69.999 and determine its decimal places we find that, like the number in Activity 8, it has 3 decimal places. Therefore, we will have to round off a third digit, 9. Since, in this case, the digit 9 is greater than 5 then we will round it up by adding 1 to the previous digit as shown below.

Therefore, $69.999 = 70.00$, to 2 decimal places

You may in other instances be given a common fraction to round off instead of a decimal fraction as we have done above. If a common fraction is given, we have to convert it first to decimal fraction before we round it off to a given decimal places. Let us look at the following example to help you understand this

Example 1: Let us express $\frac{2}{3}$ as the decimal fraction and give the answer correct to 3 decimal places.

By using a calculator we find:

$$\frac{2}{3} = 0.6666 \dots$$

The decimal fraction 0.666 is a recurring decimal. **The recurring decimal is the decimal fraction which does not end.**

Since the digit after the third one is 6 and is greater than 5, then we will add 1 to the previous digit. The previous digit is 6 and when 1 is added to it, it will become 7.

Therefore, $0.666\dots = \mathbf{0.667}$, to 3 decimal places

Now do the following activity. This activity will help you to understand better how to change the common fraction and then round it off to the given number of decimal places.



Activity 10

Change the fraction $\frac{7}{11}$ to the decimal fraction and give your answer correct to 3 decimal places. Write down your answer and working in the space provided below.



Feedback

Feedback

By using a calculator we found

$$\frac{7}{11} = 0.636363 \dots$$

*The decimal fraction 0.636363... is also a **recurring decimal**.*

Since the fourth digit is 3 and is less than 5 then the previous digit will be left as it is.

Therefore, $0.636363\dots = \mathbf{0.636}$, to 3 decimal places

The next example is about the mixed numbers. The mixed numbers are not totally new to you since you learned about them in grade 10. The mixed numbers are numbers with both the whole numbers and common fractions.

Example 2: Let us convert $76\frac{1}{7}$ to decimal fraction giving the answer to 3 decimal places.

Since $76\frac{1}{7}$ is a mixed number, it can be written as $76 + \frac{1}{7}$. The fraction $\frac{1}{7}$ can be changed to a decimal fraction by using a calculator as:

$$\frac{1}{7} = 0.142857\dots$$

By expressing it as a decimal number we write as

$$76\frac{1}{7} = 76 + 0.142857\dots$$

$$76\frac{1}{7} = 76.142857\dots$$

Now let us round off $76.142857\dots$ to 3 decimal places. The fourth digit after decimal point is 8. Since the digit 8 is greater than 5 then we will add 1 to the previous digit.

Therefore, $76.142857\dots = \mathbf{76.143}$, to 3 decimal places

Let us do another example. In this example we will first carry out calculation and then round off the answer to 2 decimal places.

Example 3: Let us evaluate the statement 0.25×83.4 and leave the answer to 2 decimal places.

$$\begin{array}{r}
 0.25 \quad \dots\dots (2 \text{ decimal places}) \\
 \times 83.4 \quad \dots\dots (1 \text{ decimal place}) \\
 \hline
 100 \quad \text{---} \boxed{25 \times 4} \\
 750 \quad \text{---} \boxed{25 \times 3} \\
 + 20000 \quad \text{---} \boxed{25 \times 8} \\
 \hline
 20.850 \quad (3 \text{ decimal places})
 \end{array}$$

Now, when 20.850 is rounded off to 2 decimal places, it will be **20.85**

So far you have learned how to round off decimal numbers to a given decimal place. Now we will look at rounding off numbers to a given significant figures. Rounding off numbers to the required significant figures is important in the sense that we do not usually give the exact number whenever we are dealing with measurements.

Significant Figures

In your Junior Certificate programme you learned how to round off numbers to the required number of significant figures. You learned that significant figures are digits in a numeral that are relevant to the value of the number represented.

Let us determine the number of significant figures of the given numbers.

Determine The Number of Significant Figures

Zero can be considered a significant or not a significant digit, depending on the position it occupies in the numeral. To illustrate this fact, let us examine some numbers in which zeros are significant and numbers in which zeros are not significant. Let us begin with whole numbers in which zeros are written at the end of a number.

Zeros at The End of The Whole Number

In any given number, whenever a zero is written at the end of the number, it is considered not significant.

Example 4: Consider the number 3 450. In this number the first three numbers are non-zero digits and the last digit is zero. Therefore, the number 3 450 has 3 significant figures. The zero at the end serves to indicate place value and have no significance apart from this. I hope you still remember place value. Therefore, the place value of the zero is zero ones.



Note it!

Zeros at the end of a whole number are not significant.

Zeros Between Non-zero Digits are Significant



Activity 11

Consider the number 30 058 and state the number of significant figures it has. You are given a chance to find the answer to this problem first and then we will do it together. Write down your answer in the space provided below.



Feedback

Feedback

When you examined the number 30 058 carefully, you should have noted that there are two zeros written in between 3 and 5. Unlike zeros written at the end of the number, these zeros are regarded as significant since they indicate that they are followed by non-zero digit. Therefore, the number 30058 has five significant figures, namely 3, 0, 0, 5 and 8.



Note it!

Zeros between non-zero digits are significant.

Let us look at significant figures in decimal numbers.

Zero at the beginning of a decimal number

Activity 12

Examine the number 0.0082 and determine the number of significant figures. Write down your answer in this space.



Feedback

Feedback

The number 0.0082 has 2 significant figures. The digits 8 and 2 are significant while the first three zeros at the beginning of the number are not significant. They serve to indicate place value.



Note it!

Zeros at the beginning of a decimal number are not significant.

Now let us look at zeros at the end of decimal numbers

Zeros at The End of a Decimal Number



Activity 13

Study the number 26.040 carefully and determine the number of significant figures it has. Write down your answer in the space below.

Feedback



Feedback

The number 26.040 has 5 significant figures. The zero here is significant as it serves to indicate that the number is greater than 26.039 and less than 26.041



Note it!

Zeros at the end of a decimal number are significant.

Now let us discuss how to write numbers correct to the required number of significant places.

Writing Numbers to a Given Number of Significant Figures

Consider the whole number 4 946. You should notice that this number has 4 significant figures. We can round off this number to any number of significant figures.



Activity 14

Let us express the number 4 946 to one significant figure. Write down the answer in the space provided below.



Feedback

Feedback

The first digit in the number 4 946 is considered significant. Therefore, the other digits have to be rounded off. The second digit is 9 and is greater than 5. Therefore, we round up 9 up by adding 1 to the previous digit, 4, to get 5 and substitute zeros for the other digits.

Thus, 4 946 = 5 000, to 1 significant figure.

Let us look at an example in which we have to carry out some calculations first and then round off the answer to a given number of significant figures.

Example 5: If one egg- yolk has the same amount of protein as a 250 grams piece of meat. If Cecilia had 23 eggs in a month, calculate the mass of meat she could have eaten of the same amount of protein, giving your answer in kilogram, round off to two significant figures.

Solution

This problem can be worked out as follows:

$$\begin{aligned} 1 \text{ egg - yolk} &= 250 \text{ g of meat} \\ 23 \text{ egg-yolks} &= x \\ \text{Therefore, mass of meat} &= 23 \times 250 \text{ g} \\ &= 5750 \text{ g} \end{aligned}$$

We use cross multiplication to come up with an equation

We can as well convert 5750g to kilogram by dividing it by 1000 as follows:

Note that 1000g = 1 kg.

$$\text{Therefore, } 5750 \text{ g} = \frac{5750}{1000} \text{ kg}$$

$$= 5.75 \text{ kg}$$

$$= \mathbf{5.8 \text{ kg, to 2 significant figures}}$$

You have come to the end of Topic 1. We hope that you enjoyed working through this topic and you felt motivated to take part in doing the activities in order to strengthen your skills in applying some of the new knowledge which you acquired from our

discussions. Now we would like to invite you to read through a topic summary that follows.

Topic 1: Summary



Summary

In this first topic of unit 1: approximation and scientific notation, you have learned how to round off numbers to the nearest unit and fraction of a unit. You also learned to express numbers correct to the required number of decimal places and required number of significant figures.

We also noted that:

- Zeros written in between non-zero digits are significant as in 2004 because they indicate that they are followed by a non-zero digit.
- Zeros written at the end of non-zero digits on a whole number are not significant as in 2400 because they indicate place value.
- Zeros written at the beginning of non-zero digits in a decimal number are not significant as in 0.0024 because they indicate place value.
- Zeros written at the end of a decimal number are significant as in 0.02400 because they indicate that the number is greater than 0.02399 and less than 0.02401.

You were also encouraged to answer all the questions in the activities of this topic. The activities are intended to help you assess how well you understood the content of the various sub-sections of the topic. If you did not do very well in the activity, this means that you need to go over the material again.

In this topic we discussed that some quantities and measures can be approximated to a given degree of accuracy. We also discussed that numbers can be corrected to the required number of decimal places and significant figures. In the next topic we will discuss how these numbers can be expressed in scientific notation.

Now you should do topic exercise 1 that comes soon after topic 3 to see how much you have learned in the whole topic. After completing the exercise, you should mark your own work by comparing your answers with those provided at the end of this unit.

Topic 2: Scientific Notation (Standard Form)

Sometimes we deal with very big numbers like the speed of light or very small numbers like the mass of an atom. These numbers are not easily read out due to several digits they have. One way of making these big or small numbers easily written is by expressing them in scientific notation or standard form. Therefore, in this topic, you will learn how to express numbers in scientific notation. Scientific notation is one of the interesting topics in mathematics course in the sense that it helps us to write big or small numbers in a way we can read or write them easily. Scientific notation is not new to you since you learned it in your Junior Certificate Programme.

After studying through the topic you will, as was the case in topic 1, be required to work out topic 2 exercise and mark your work using the feedback provided immediately. You are again encouraged not to go through the feedback before doing the exercise.

In this second topic of the unit, we will address the fifth of the seven unit outcomes. This means that we have only one outcome for this unit.

Upon completion of this topic you will be able to:

- Write numbers in scientific notation (Standard form).



Outcomes

Defining Scientific Notation

Scientists and engineers usually deal with very large numbers or very small numbers, for example the chemists, are able to determine the number of molecules of any given gas that occupies the volume of 24 dm^3 as 600 000 000 000 000 000 000 000 molecules. As you can see this is very a large number. On the other hand they are able to determine the mass of an atom of hydrogen as 0.000 000 000 000 000 000 000 001 66 g. You can

also notice that this is very small number. These numbers can be read properly if they are written in a scientific notation.

Please you should note that writing numbers in scientific notation is not new to you. You learned about it in your Junior Certificate Programme. If you remember properly you learned that scientific notation is a way of expressing a given number as a number between 1 and 10 multiplied by 10 to the appropriate power. Do you still remember what power is in this case? Of course, the other name of power is **index**, which is a number written above 10. You should note that scientific notation is also called **standard form**. Therefore, we will be using both terms in this unit.

The number in scientific notation can be written as:

$A \times 10^n$ where

- A is a number between 1 and 10 or $1 \leq A < 10$, and
- n is a negative or positive integer or $n \in \mathbb{Z}$

You should note that the symbols we have used here are:

- $<$ “which stands for “is less than”,
- \leq stands for “is less than or equal to”,
- \in stands for “is a member of”, and
- \mathbb{Z} which stands for “is a set of integers”.

These symbols are not new to you since you learned about them in grade 10. The word integer is not new to you either. You learned that integer is the set of whole numbers, negative numbers and zero.

Now you will be taken through step by step on how to write numbers in scientific notation for both numbers greater than 1 and numbers less than 1. We will start with numbers that are greater than 1.

Numbers Greater Than 1

Consider the number 42 073.8. This number has the whole number (five digits written before a decimal point) and the decimal part (a digit written after decimal point). This number can be written to the power of ten as follows:

$$42073.8 = 4.20738 \times 10000$$

Please note that the first number we have just written is less than ten but greater than one.

The last number 10000 can be written in index form as follows:

$$42073.8 = 4.20738 \times 10 \times 10 \times 10 \times 10$$

$$42073.8 = 4.20738 \times 10^4$$

This 4 represents the number of times 10 is multiplying itself

Sometimes you may be required to carry out calculations before expressing the number in standard form. Consider the example below.

Example 1: Supposing the number of students infected with HIV at a certain college makes up two thirds of the student population. If there are 1 680 students infected with HIV at the college, how many students are there at the college altogether? Express your answer in standard form.

Solution

Since the total number of students is not known, then we express the unknown number as x .

Firstly, we may be required to find the total number of students at the college. This can be worked out as follows

Let x be the total number of students.

So we will have $\frac{2}{3}$ of $x = 1680$.

We can, as well, use the multiplication sign to express the above statement as shown below. You should note that the word “of” appearing in between $\frac{2}{3}$ and x in the above mathematical statement, in mathematics, it means multiplication sign (\times):

$$\text{Total number of students} = \frac{2}{3} \times x = 1680.$$

Note that "of" has been changed to multiplication sign (\times)

The equation can be simplified further as follows.

$$\frac{2x}{3} = 1680$$

Then we will multiply both sides of the equation by the reciprocal of $\frac{2}{3}$. Reciprocal is a number that is related to another by the fact that when multiplied together the product is one.

The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$ since $\frac{2}{3} \times \frac{3}{2} = \frac{2 \times 3}{3 \times 2} = \frac{6}{6} = 1$.

And,

$$\begin{aligned}\frac{2x}{3} \times \frac{3}{2} &= 1680 \times \frac{3}{2} \\ \frac{6x}{6} &= \frac{5040}{2} \\ x &= 2520\end{aligned}$$

Simplify the fractions in their simplest form

Therefore, there are 2520 students in the college.

Then we have to express the number of students at the college in standard form. This can be shown as follows:

The number 2520 can be expressed as the power of ten as;

$$2520 = 2.520 \times 1000$$

$$2520 = 2.520 \times 10^3$$

Remember that
 $1000 = 10 \times 10 \times 10$
 $= 10^3$

So far you have learned how to express numbers in standard form or scientific notation which is expressed as

$$A \times 10^n \text{ where } 1 \leq A < 10 \text{ and } n \in \mathbb{Z}$$

The numbers written in standard form can be reversed or evaluated. Let us go through the following example.

Example 2: The number 2.8×10^3 can be evaluated as follows:

$$2.8 \times 10^3 = 2.8 \times 10 \times 10 \times 10$$

$$= 2.8 \times 1000$$

$$= 2800.0$$

$$= \mathbf{2800}$$

Express 10^3 in expanded form

Add three zeros to 2.8. Since the numbers that are been multiplied have one decimal place, then correct the product to one decimal place also.

When two or more numbers written in standard form are multiplying, brackets are used to separate them. When simplifying

the expression, we remove the brackets first and then we collect the like terms. The following example will help you understand how to multiply numbers written in standard form.

Example 3: The expression $(2.2 \times 10^5) \times (2 \times 10^8)$ can be worked out as follows:

$$\begin{aligned} &(2.2 \times 10^5) \times (2 \times 10^8) \\ &= 2.2 \times 10^5 \times 2 \times 10^8 \\ &= 2.2 \times 2 \times 10^5 \times 10^8 \\ &= 4.4 \times 10^{5+8} \\ &= 4.4 \times 10^{13} \\ &= 44\,000\,000\,000\,000.0 \\ &= 44\,000\,000\,000\,000 \end{aligned}$$

We will rearrange numbers without powers together and those with powers together.

According to the law of indices, we add powers when multiplying numbers of the same bases

Since the power of ten is 13, then we will add 13 zeros to 44. The answer will be written to one decimal place since the original number has one decimal place.

In our example above we have used the addition law of indices to simplify numbers written in standard form. The laws of indices are not new to you since you learned about them in grade 10.



Note it!

Using addition law of indices, we added the powers of the numbers whose bases are the same.

$$a^m \times a^n = a^{m+n} \text{ where } m \text{ and } n \text{ are negative or positive integers}$$

The activity you are going to do now is similar to the example you have just learned. You will be required to use the knowledge of significant figures you learned in Topic 1.



Activity 1

In the space provided below, evaluate $(5.548 \times 10^5) \times (6.2 \times 10^{-3})$, leaving your answers in standard form and round it off to 3 significant figures

Feedback

Feedback

We hope the examples we worked out earlier enabled you to complete this activity with relative ease. If after reading our feedback you discover that your answer was incorrect, revise this part of your work to improve your understanding.

In order to evaluate this problem we have to identify the numbers with powers and numbers without powers. Then we have to rearrange them. We can as well use brackets to group them as shown below:

$$(5.548 \times 10^5) \times (6.2 \times 10^{-3})$$

$$= 5.548 \times 10^5 \times 6.2 \times 10^{-3}$$

$$= (5.548 \times 6.2) \times (10^5 \times 10^{-3})$$

$$= 34.3976 \times 10^{5+(-3)}$$

$$= 34.3976 \times 10^{5-3}$$

$$= 34.3976 \times 10^2$$

$$= 3.43976 \times 10^1 \times 10^2$$

$$= 3.43976 \times 10^{1+2}$$

$$= 3.43976 \times 10^3$$

$$= \mathbf{3.44 \times 10^3}$$

We will work out what is in the brackets first.

We multiply numbers with same base by adding their powers

You should note that the number 34.3976 is greater than 10. We will therefore write it in standard form as 3.43976×10^1

We will finally round off the answer to 3 significant figures.

Like the activity we have just done, the numbers written in standard form can be divided. The following worked example shows step by step how division of numbers written in standard form are worked out.

Example 4: Let us simplify $(9.6 \times 10) \div (1.2 \times 10^{-2})$

To simplify this expression we will rearrange numbers without powers together and those with powers together and divide them.

$$(9.6 \times 10^1) \div (1.2 \times 10^{-2})$$

$$= 9.6 \div 1.2 \times 10^{-1} \div 10^{-2}$$

We remove the brackets and rearrange the numbers.

$$= (9.6 \div 1.2) \times (10^{-1} \div 10^{-2})$$

$$= 9.6 \div 1.2 \times 10^{1-(-2)}$$

$$= 9.6 \div 1.2 \times 10^{1+2}$$

$$= 9.6 \div 1.2 \times 10^3$$

$$= 8 \times 10^3$$

$$= \mathbf{8000}$$

We subtract powers of the numbers with same base.

Multiplication of two negative signs gives a positive sign. That is $- \times - = +$

You should note that this time we have used the subtraction law of indices to divided numbers written in standard form. Remember you learned about this law in grade 10.



Note it!

Using subtraction law of indices, we subtract the powers of the numbers whose bases are the same.

$$a^m \div a^n = a^{m-n} \text{ where } m \text{ and } n \text{ are negative or positive integers}$$

The activity you are going to do now is similar to the example you have just learned. You will be required to use the knowledge of significant figures you learned in Topic 1.



Activity 2

Evaluate $1.2 \times 10^6 \div 8.2 \times 10^2$, leaving your answers in standard form and collect to 3 significant figures. Write down your answer and working in the space provided below.



Feedback

Feedback

The problem in Activity 11 can be worked out as follows:

$$\begin{aligned}
 &1.2 \times 10^6 \div 8.2 \times 10^2 \\
 &= 1.2 \times 10^1 \times 10^5 \div 8.2 \times 10^2 \\
 &= 12 \times 10^5 \div 8.2 \times 10^2 \\
 &= (12 \div 8.2) \times (10^5 \div 10^2) \\
 &= 12 \div 8.2 \times 10^{5-2} \\
 &= 1.4634 \times 10^3 \\
 &= \mathbf{1.46 \times 10^3}
 \end{aligned}$$

Before we rearrange the numbers, you should note that the dividend 1.2 is smaller than the divisor 8.2. We will therefore increase the dividend by multiplying it by 10 from 10^6

Work out what is in the bracket first

We will finally round off the answer to 3 significant figures.

Did you get the answer right? If necessary revise the section to ensure that you build a solid foundation to all these aspects of scientific notation.

The section you are going to study now is similar to the one you have just studied. This time we will deal with numbers that are less than 1.

Numbers Less Than 1

You might have noticed that when we express numbers greater than 1 the power of 10 is always a positive number. Now we will deal with numbers whose power of 10 is a negative number. These numbers are less than 1.

Let us look at the following example together.

Example 5: Consider the number 0.0006. This number can be expressed in standard form as follows:

First we have to express 0.0006 as a fraction.

The first step we need to do is to determine the place value of a non-zero digit in the number.

ones	Decimal point	Tenth	Hundredth	Thousandth	Ten thousandth
------	---------------	-------	-----------	------------	-----------------------

0	.	0	0	0	6
---	---	---	---	---	---

You can notice from the table above that the number 6 is ten thousandths. Please note that 6 ten thousandths is written as

$$\frac{6}{10\,000}$$

Note that the denominator is ten thousand.

We can therefore write 0.0006 as a fraction as follows:

$$0.0006 = \frac{6}{10000}$$

In order to write $\frac{6}{10000}$ in the form $A \times 10^n$ where $1 \leq A < 10$ and $n \in \mathbb{Z}$, we have to use the laws of indices you learned in Grade 10.

$$\begin{aligned} \frac{6}{10000} &= \frac{6}{10 \times 10 \times 10 \times 10} \\ &= \frac{6}{10^4} \\ &= 6 \times 10^{-4} \end{aligned}$$

We first write 10000 in expanded form to determine the power of ten

Note that when we multiply the numerator by its denominator the sign of the power changes.

The activity you will do now is similar to the example you have just studied.



Activity 3

Write the number 0.0085 in standard form. Use the space provided below to write down your answer and working.



Feedback

Feedback

The number 0.0085 can be written in standard form as follows:

The 0.0085 should be written in the form

$A \times 10^n$ where $1 \leq A < 10$ and $n \in \mathbb{Z}$,

$$\begin{aligned} 0.0085 &= \frac{8.5}{1000} \\ &= \frac{8.5}{10 \times 10 \times 10} \\ &= \frac{8.5}{10^3} \\ &= 8.5 \times 10^{-3} \end{aligned}$$

Decimal fraction is inserted soon after 8.

We can use the short method to write the numbers less than 1 in standard form. In this method you just count the number of decimal places the decimal point has moved in order to make the number take the form $A \times 10^n$ where $1 \leq A < 10$ and $n \in \mathbb{Z}$,

Let us use the example we earlier discussed.

We learned that the number

$$\begin{aligned} 0.0006 &= \frac{6}{10000} \\ &= \frac{6}{10 \times 10 \times 10 \times 10} \\ &= \frac{6}{10^4} \\ &= 6 \times 10^{-4} \end{aligned}$$

From this example you should note that the power of 10 corresponds to the number of digits the decimal point moves through to its new position as shown below. You should remember that you have already done this in relation to positive numbers. The process is now reversed.



Thus, it becomes 6.0×10^{-4}

Note that we usually omit “point zero, (.0)” at the end of the whole numbers. Therefore, we would write this expression as follows

$$0.0006 = 6 \times 10^{-4}$$

You should note that the power of 10 has a negative sign. Do you know why? To answer this question let us consider the following aspects:

Whenever the decimal point moves to the right it carries a negative sign. On the other hand whenever the decimal point moves to the left it carries a positive sign. Note that we will use this approach frequently in our calculations since it is regarded as a short method of determining the power of 10 when expressing the number in standard form.

We can as well deduce why the power of 10 has a negative sign by multiplying a^{-m} by a^m using the addition law of indices as shown below.

$$\begin{aligned} a^{-m} \times a^m &= a^{-m+m} \\ &= a^0 \end{aligned}$$

Since the powers have different signs, when we simplify them we get zero

Note that any non-zero number raised to power 0 gives 1 as shown below.

$$a^0 \times a^m = a^{0+m}$$

$$a^0 \times a^m = a^m$$

$$\begin{aligned} \frac{a^0 \times a^m}{a^m} &= \frac{a^m}{a^m} \\ a^0 &= 1 \end{aligned}$$

You should note that the sum of “0” and “m” gives “m”

To express the equation in terms of a^0 , we have to divide both side of the equation by a^m

Therefore,

Since $a^{-m} \times a^m = a^0$, then

$$a^{-m} \times a^m = 1$$

By dividing both sides of the equation by a^m , we have

$$\frac{a^{-m} \times a^m}{a^m} = \frac{1}{a^m}$$

$$a^{-m} = \frac{1}{a^m}$$

We simplify the equation

Now you should do an activity. The activity you are going to do is similar to the example we have just discussed.

Consider the number 0.00152. Express it in standard form. Use the space below to write your answer and working.



Activity 4



Feedback

Feedback

Remember the number 0.00152 should be expressed in the form:

$$A \times 10^n \text{ where } 1 \leq A < 10 \text{ and } n \in \mathbb{Z}$$

Therefore, we have to move the decimal point from its current position and insert it in between 1 and 5 so that the first number in the expression should be greater than 1 but less than 10. You might have noticed that the decimal point has moved through three decimal places to the right. Therefore, the number can be written in standard form as follows:

$$\begin{aligned} 0.00152 &= \frac{1.52}{1000} \\ &= \frac{1.52}{10 \times 10 \times 10} \\ &= \frac{1.52}{10^3} \\ &= 1.52 \times 10^{-3} \end{aligned}$$

We have to move the decimal point and insert it soon after 1.

We have to determine the value of the power of 10

The following activity is similar to the previous one. You should express the number in standard form using the short method you have just learned.



Activity 5

The wavelength of a wave is 0.0000018 cm. Express the wavelength in standard form in centimetres. Write down your answer and work in the space provided below.



Feedback

Feedback

The number 0.0000018 cm can be written in standard form as follows:

Remember that we have to write this number in the form $A \times 10^n$. Since the decimal point moves 6 decimal places to the right then the power of 10 will be -6.

Therefore, $0.0000018 = 1.8 \times 10^{-6} \text{ cm}$

The activity you are about to do is similar to the one we have just looked at. However, this time you will be required to express the answer to 3 significant figures. You should remember that you learned about significant figures in Topic 1 of this unit and you may revise relevant part of Topic 1 if necessary before completing this activity.



Activity 6

Express 0.0012301 in standard form and correct it to 3 significant figures. Write down your answer and work in the space provided below.

Feedback

Feedback

The number 0.0012301 can be written in standard form as

$$\begin{aligned} 0.0012301 &= 1.2301 \times 10^{-3} \\ &= 1.23 \times 10^{-3}, \text{ correct to 3 significant figures.} \end{aligned}$$

You should note that the decimal point has moved three decimal places to the right to make the first factor greater than 1 but less than 10. You should also note that the other factor, 10, is raised to the power of -3.

The next activity you will do is similar to the one we have just discussed. You will be required to round off the answer to the 3 significant figures.



Activity 7

Express the number 0.300565 in standard form. Write down your answer and work in the space below.

Feedback



Feedback

The number 0.300565 can be written in standard form as follows:

$$\begin{aligned}0.300565 &= 3.00565 \times 10^{-1} \\ &= \mathbf{3.01 \times 10^{-1}}, \text{ correct to 3 significant figures.}\end{aligned}$$

You should note that decimal point has moved one decimal place to the right to make the first factor greater than 1 but less than 10 and the power of 10 is -1.

We have come to the end of Topic 2. You should read the topic summary that follows.

Topic 2: Summary



Summary

In the second topic of Unit 1: approximation and scientific notation, you have learned how to express numbers in scientific notation.

The number in scientific notation can be written as:

$A \times 10^n$ where

- A is a number between 1 and 10 or $1 \leq A < 10$, and
- n is a negative or positive integer or $n \in \mathbb{Z}$

You also learned that

- the power of 10 for numbers greater than 1 is always a positive integer as in $456000 = 4.56 \times 10^5$
- the power of 10 for numbers less than 1 is always a negative integer as in $0.000123 = 1.23 \times 10^{-4}$

Now you should do the following topic exercise. The exercise is intended to help you assess how well you understood the content of the topic. We encourage you to answer all the questions in the topic exercise. After completing the exercise, you should mark your own work by comparing your answers and work with those provided in the feedback that follows. If you did not do very well in the exercise, this means that you need to go over the material again.

The next topic is about estimation of error. In this topic we will discuss how to determine the extent to which we can rely on our measurements.

Topic 3: Estimation of Error

This is the last topic in this unit: approximations and scientific notation. Estimation of error is another of the interesting topics in mathematics. Like the two topics we discussed above, estimation of error is not totally new to you in the sense that you learned about it in your Junior Certificate Programme. This time you will learn different type of errors and how to calculate them. You will be taken through the calculations step by step. Estimation of error is important in life and is widely used in manufacturing industry and fields like building and engineering. This makes it to be one of the important topics in mathematics

The errors we will discuss are absolute error, relative error and percentage error. You will further learn how to use approximations to estimates of error. You will learn how to find the acceptable measurements for the given number, the tolerance, the maximum and minimum sum, difference, area and mean of the measurements

After studying through the topic you will be required to work out topic exercises. There are 2 topic exercises in this topic for you to assess your own progress. These are topic exercises 3 and 4. You should mark your work using the feedback provided immediately. As in the first two topics, you are encouraged not to go through the feedback before doing the exercises.

In this last topic of the unit, we will address the last two of the seven unit outcomes.

Upon completion of this topic you will be able to:

- Calculate estimates of error.
- Use approximations to estimates of error.



Outcomes

Defining Estimation of Error

To estimate is to approximate measurements or numbers. We earlier discussed in topic 1 that a measurement can never be exact.

No matter how careful we are in carrying out a measurement, we can never know what the correct measure is. You applied this rule in topic 1, activity 2 when you and your two friends measured the height of the triangle and hopefully found that your measurements differed. When we are asked to draw a line of 20 cm long, for example, we might not draw a line of exactly 20 cm long. We may either draw a line slightly more than 20 cm or less than 20 cm. The difference between this true measurement and the one obtained by measurement is called an **error**, even though we have made no mistake in the actual measurement.

When we measure physical quantities there is always some degree of error or uncertainty. Errors are caused by the limitations of the measuring instrument, the conditions under which the measurement is made, and the different ways the operator uses the instrument. The size of this error can be reduced by using more accurate instruments.

As we have earlier discussed that errors in measurements can never be exact, therefore errors can never be completely eliminated. It is most important that we should know in any situation the extent to which we can rely on our measurements, that is, we should know the maximum possible error involved. In the topic we discuss absolute error, relative error and percentage error.

The Absolute Error

The absolute error is one of the errors that we make whenever we are taking measurements. The term is new to you since this is the first time we are introducing it to you in your mathematics course. However, we will use some illustrations to help you understand its meaning better.

Consider the measurement of a straight line. If we use a ruler graduated in centimetres, we can determine the length of the given distance to the nearest tenth of a centimetre. For example, we may give the length of a line as 5.2 cm. In this case we have estimated the length by giving the length correct to the nearest 0.1 cm, and we say that the **least unit of measurement** is 0.1 cm. The error may be as much as 0.05 cm. That is, it may lie anywhere between 5.15 cm ($= 5.2 - 0.05$ cm) and 5.25 cm ($= 5.2$ cm + 0.05 cm)

In this case the error, 0.05 cm, is referred to as an **absolute error**.

From the illustration we have just discussed we can conclude that the absolute error is half the least unit of measurement. That is, the least unit of measurement is 0.1 cm and the absolute error is 0.05 cm.



Note it!

Absolute error is half the least unit of measurement. For example, the least unit of measurement for a ruler graduated in centimetres is 0.1 cm and its absolute error is 0.05 cm ($0.1 \div 2$)

The next illustration you will look at is similar to the one you have just studied. This time we will use a different unit of measurement.

If someone tells you that the mass of an object is 1.76 g, this does not imply that the mass is exactly 1.76 g. This measurement is just an approximation. When the measurement was taken, the reading might have been slightly over or less than 1.76 g. The measurement was rounded off to the nearest hundredths gram, which is 0.01 g.

Thus the true mass is near 1.76 g than to 1.75 g or 1.77 g; i.e. it may lie anywhere between 1.755 g and 1.765 g.

In this case the error may be as much as 0.005 g. That is

$$\text{Upper limit} = 1.76 \text{ g} + 0.005 \text{ g} = 1.765 \text{ g}$$

$$\text{Lower limit} = 1.76 \text{ g} - 0.005 \text{ g} = 1.755 \text{ g}$$

Therefore, the **absolute error** is 0.005 g.

In the sections that follow you will do four activities to apply and consolidate your knowledge of absolute error. The first activity you are about to do is similar to the illustrations you have just studied. The activity will help you to understand how to find the absolute error.



Activity 1

Consider a mass 15 kg, correct to the nearest kg. Find its absolute error. Write down your answer and work in the space provided below.



Feedback

Feedback

Was it easy to work out? Remember, to find the absolute error of 15 kg, we must first find the least unit of measurement. From the question, we have been told that the least unit of measurement of 15 kg is 1 kg.

Secondly, we have to find the absolute error by multiplying the least unit of measurement by $\frac{1}{2}$

$$\begin{aligned} \text{Therefore, absolute error} &= 1 \text{ kg} \times \frac{1}{2} \\ &= \mathbf{0.5 \text{ kg}} \end{aligned}$$

Therefore, the absolute error of 15 kg is 0.5 kg.

As we earlier discussed in this topic, we can use the absolute error to find the upper limit and lower limit of measurements. Therefore, the following activity will help you understand better how to calculate the upper limit and lower limit of measurement.



Activity 2

Find the upper limit and lower limit of 7 cm. Write down your answer and work in the space provided below.



Feedback

Feedback

The first step is to find the least unit of measurement.

The least unit of measure of 7 cm is 1cm

*Absolute error is $\frac{1\text{cm}}{2}$ of 1cm = **0.5 cm***

*Upper limit is = 7 + 0.5 = **7.5 cm***

*Lower limit is = 7 - 0.5 = **6.5 cm***

Now find the absolute error

We now move on to activity 3. This activity which you are about to do is similar to the one you have just done. Note that this time the least unit of measurement is not given in the question. You have to find it on your own.



Activity 3

Find the upper limit and lower limit of 5.4 cm. Write down your answer and work in the space provided below.



Feedback

Feedback

The upper limit and lower limit of 5.4 cm can be worked as follows:

The least unit of measure of 5.4 cm is 0.1cm

$$\text{Absolute error is} = \frac{0.1 \text{ cm}}{2} = \mathbf{0.05 \text{ cm}}$$

$$\begin{aligned} \text{Upper limit} &= 5.4 + 0.05 \\ &= \mathbf{5.45 \text{ cm}} \end{aligned}$$

$$\text{Lower limit} = 5.4 - 0.05$$

The last activity in this section will help you understand better how to find the upper limit and lower limit of measurement. The activity is similar to the one you have just done, but this time the measurement is rounded off to two decimal places.



Activity 4

Find the upper limit and lower limit of 14.36 km. Write down your answer and work in this space.



Feedback

Feedback

The upper limit and lower limit of 14.36 km can be worked out as follows:

The least unit of measure of 14.36km is 0.01km.

$$\begin{aligned} \text{Absolute error} &= \frac{0.01 \text{ km}}{2} \\ &= \mathbf{0.005 \text{ km}} \end{aligned}$$

$$\begin{aligned} \text{Upper limit} &= 14.36 + 0.005 \\ &= \mathbf{14.365 \text{ km}} \end{aligned}$$

$$\begin{aligned} \text{Lower limit} &= 14.36 - 0.005 \\ &= \mathbf{14.355 \text{ km}} \end{aligned}$$

The above four activities and the feedback provided should have helped you to understand absolute error. If you got some of the activities wrong, revise the examples and redo the activities. You should then discuss any points of difficulty with other learners and/or your tutor where possible.

Another type of error we will look at is relative error. When calculating relative error we compare the absolute error with true measurement. This comparison of numbers is referred to as the ratio. Calculation of ratio is not new to you since you learned about it in your Junior Certificate Programme. Therefore, relative error is the ratio of the absolute error in a measurement to the size of the measurement. Relative error is often expressed as a percentage and is called the **percentage error**

Relative Error and Percentage Error

Errors made when demarcating or setting the boundaries of a piece of land can be ignored. But errors made by engineers are very serious and can cost them a lot or damage the work. Relative error can be worked as follows:

$$\text{Relative error} = \frac{\text{absolute error}}{\text{true measurement}}$$

Let us go through the following example together. It will help you to understand how to find the relative error and percentage error.

Example 1: Let us find the relative error and the percentage error of 9.7 cm together.

Solution

The relative error and the percentage error of 9.7 cm can be worked as:

The least unit of measurement of 9.7 cm is 0.1cm.

$$\text{Absolute error} = \frac{0.1 \text{ cm}}{2} = 0.05 \text{ cm}$$

$$\text{Relative error is} = \frac{0.05 \text{ cm}}{9.7 \text{ cm}}$$

To simplify this fraction to its simplest form, we have to make its numerator and denominator into whole numbers by multiplying them by 100.

$$\begin{aligned} \text{Relative error} &= \frac{0.05 \times 100}{9.7 \times 100} \\ &= \frac{5}{970} \\ &= \frac{1}{194} \end{aligned}$$

If we wish to express this as a percentage, we can multiply by 100%.

$$\text{Percentage error} = \frac{1}{194} \times 100\%$$

$$= \mathbf{0.515\%} \text{ (correct to 3 significant figures)}$$

From the above example, you should now be able to find relative error. Do activity 5 below which is similar to the example you have just studied.



Activity 5

Find the relative error, and the percentage error of 7.45 l to 2 significant figures. Write down your answer and work in the space provided below.



Feedback

Feedback

Were you able to find relative error? Let us take you through the steps for doing this and then compare our answer with yours.

The relative error of 7.45 l can be worked out as follows:

The least unit of measure of 7.45 l is 0.01l.

$$\text{Absolute error} = \frac{0.01}{2} = 0.005 \text{ l}$$

$$\begin{aligned} \text{Relative error is} &= \frac{0.005}{7.45} \\ &= \frac{0.005 \times 1000}{7.45 \times 1000} \\ &= \frac{5}{7450} \\ &= \frac{1}{7450} \end{aligned}$$

$$\begin{aligned} \text{Percentage error} &= \frac{1}{7450} \times 100\% \\ &= \mathbf{0.013\%} \end{aligned}$$

Was your answer different? Check where you went wrong and make the necessary corrections.

The next activity you are about to do is similar to the one you have just done. This time you will be required to round off your answer to 2 significant figures.



Activity 6

Find the relative error, and the percentage error of 58.6 mm to 2 significant figures. Write down your answer and working in the space provided below.



Feedback

Feedback

The relative error of 58.6 mm can be worked out as follows:

The least unit of measure of 58.6 mm is 0.1 mm

$$\text{Absolute error} = \frac{0.1}{2} = 0.05 \text{ mm}$$

$$\begin{aligned} \text{Relative error is} &= \frac{0.05}{58.6} \\ &= \frac{0.05 \times 100}{58.6 \times 100} \\ &= \frac{5}{5860} \\ &= \frac{1}{1172} \end{aligned}$$

Now you can simplify the fraction by dividing both numerator and denominator by 5.

$$\text{Percentage error} = \frac{1}{1172} \times 100\%$$

$$= 0.085324\%$$

$$= \mathbf{0.085\%, \text{ correct to 2 significant figures}}$$



Note it!

The smaller the least unit of value the smaller the relative error, and when the error is small then the measurement is more accurate.

So far you have learned how to find the absolute error, relative error and percentage error. Now you can do activity 7. This activity

is longer than the ones you did earlier and will enable you to assess how much you have learned about absolute error, relative error, and percentage error. After completing the activity, you must mark your work by comparing your answers and working with the feedback provided.



Activity 7

There are 4 questions in this activity. Answer all the questions. Write your answers and working on a separate paper.

1. Find the absolute error of the following :
 - (a) 19 Kg
 - (b) 12.03 m
 - (c) 0.005 km

2. Find the upper limit and lower limit of each of these figures.
 - (a) 17 kg
 - (b) 15.04 m
 - (c) 30°

3. Find the relative error in each of the following
 - (a) 174m
 - (b) 24 seconds

4. In each of the following, find the relative error and percentage error to 3 significant figures.
 - (a) 8 cm
 - (b) 140 kg

Now you can compare your answers and work with those provided in the feedback below.



Feedback

Feedback

1. (a) The least unit of measurement of 19 Kg is 1 kg

$$\text{Absolute error} = \frac{1}{2} \times 1 \text{ kg} = \mathbf{0.5 \text{ kg}}$$

- (b) The least unit of measurement of 12.03 m is 0.01m

$$\text{Absolute error} = \frac{1}{2} \times 0.01 \text{ m} = \mathbf{0.005 \text{ m}}$$

- (c) The least unit of measurement of 0.005 km is 0.001 km

$$\text{Absolute error} = \frac{1}{2} \times 0.001 \text{ km} = \mathbf{0.0005 \text{ km}}$$

2. (a) The least unit of measurement of 17 kg is 1 kg.

$$\text{Absolute error} = \frac{1 \text{ kg}}{2} = 0.5 \text{ kg}$$

$$\begin{aligned} \text{Upper limit} &= 17 + 0.5 \\ &= \mathbf{17.5 \text{ kg}} \end{aligned}$$

$$\begin{aligned} \text{Lower limit} &= 17 - 0.5 \\ &= \mathbf{16.5 \text{ kg}} \end{aligned}$$

- (b) The least unit of measurement of 15.04 m is 0.01 m.

$$\text{Absolute error} = \frac{0.01 \text{ m}}{2} = 0.005 \text{ m}$$

$$\begin{aligned} \text{Upper limit} &= 15.04 + 0.005 \\ &= \mathbf{15.045 \text{ m}} \end{aligned}$$

$$\begin{aligned} \text{Lower limit} &= 15.04 - 0.005 \\ &= \mathbf{15.035 \text{ m}} \end{aligned}$$

- (c) The least unit of measurement of 30° is 1°

$$\text{Absolute error} = \frac{1^\circ}{2} = 0.5^\circ$$

$$\begin{aligned} \text{Relative error} &= \frac{0.5^\circ}{30^\circ} \\ &= \frac{0.5^\circ \times 10}{30^\circ \times 10} \\ &= \frac{5}{300} \\ &= \frac{1}{60} \end{aligned}$$

3. (a) The least unit of measurement of 174 m is 1 m.

$$\text{Absolute error} = \frac{1 \text{ m}}{2} = 0.5 \text{ m}$$

$$\begin{aligned} \text{Relative error} &= \frac{0.5 \text{ m}}{174 \text{ m}} \\ &= \frac{0.5 \text{ m} \times 10}{174 \text{ m} \times 10} \end{aligned}$$

$$= \frac{5}{1740}$$

$$= \frac{5}{348}$$

(a) The least unit of measurement of 24 seconds is 1 s

$$\text{Absolute error} = \frac{1 \text{ s}}{2} = 0.5 \text{ s}$$

$$\text{Relative error} = \frac{0.5 \text{ s}}{24 \text{ s}}$$

$$= \frac{0.5 \text{ s} \times 10}{24 \text{ s} \times 10}$$

$$= \frac{5}{240}$$

$$= \frac{1}{48}$$

4. (a) The least unit of measurement of 8 cm is 1 cm

$$\text{Absolute error} = \frac{1 \text{ cm}}{2} = 0.5 \text{ cm}$$

$$\text{Relative error} = \frac{0.5 \text{ cm}}{8 \text{ cm}}$$

$$= \frac{0.5 \text{ cm} \times 10}{8 \text{ cm} \times 10}$$

$$= \frac{5}{80}$$

$$= \frac{1}{16}$$

$$\text{Percentage error} = \frac{1}{16} \times 100\% = \mathbf{6.25\%}$$

(b) The least unit of measurement of 140 kg is 1 kg

$$\text{Absolute error} = \frac{1 \text{ kg}}{2} = 0.5 \text{ kg}$$

$$\text{Relative error} = \frac{0.5 \text{ kg}}{140 \text{ kg}}$$

$$= \frac{0.5 \text{ kg} \times 10}{140 \text{ kg} \times 10}$$

$$= \frac{5}{1400}$$

$$= \frac{1}{280}$$

$$\text{Percentage error} = \frac{1}{280} \times 100\%$$

$$= \mathbf{2.36\%} \text{ (correct to 3 significant figures)}$$

We hope that after comparing your answers with the model answers and working provided in the above feedback you might have got all the answers correct. Congratulations! If not then I suggest you try the questions you got wrong again. You may be required to read through the section again in order to gain more understanding.

So far we have discussed how to find the least unit of measurement, the absolute error, relative error, percentage error, and the upper as well as the lower limits of measurement. Now we will look at how these errors are applied to real life situations.

Application of Estimation of Error

We will study the application of estimation of error under the following sub-heading:

- Tolerance
- Sum and difference of measurements
- Product and quotient of measurements

As we have done in the earlier parts of this unit, we will assist you to understand the concepts by providing examples and solutions, ask you to apply concepts in various activities and provide feedback that shows you how the answers should have been worked out.

Tolerance

An error is allowed in most companies and in some fields like building and engineering but this should be of acceptable amount that could not affect the work too much. An acceptable difference between the actual measurement and the measurement obtained is used. The difference between the greatest and the least acceptable measurements is called **tolerance**.

The formula to find the tolerance is given as follows:

$$\text{Tolerance} = \text{upper limit} - \text{lower limit}$$

The following example will help you to understand how to find tolerance.

Example 1: Let us find the tolerance of 2 kg sugar together.

Solution

Least unit of measurement of 2 kg = 1 kg

The absolute error = $\frac{1 \text{ kg}}{2} = 0.5 \text{ kg}$

Tolerance = upper limit – lower limit

$$\begin{aligned} \text{Tolerance} &= (2 + 0.5) \text{ kg} - (2 - 0.5) \text{ kg} \\ &= 2.5 \text{ kg} - 1.5 \text{ kg} \\ &= 1 \text{ kg} \end{aligned}$$

You will notice that the next example you are going to do is similar to the one you have just studied. This time, however, we will work out the process in the reverse order.

Example 2: Let us find the tolerance of the measurement whose upper limit is 6 cm and the lower limit is 2 cm.

Solution

Tolerance = upper limit – lower limit

$$\begin{aligned} \text{Tolerance} &= 6 \text{ cm} - 2 \text{ cm} \\ &= 4 \text{ cm} \end{aligned}$$

Therefore, the measurement = **4 cm**

Let us look at another example together. The example is also similar to the one you have just studied but, this time tolerance notation is used.

Example 3: If the measurement lies between $(6.3 \pm 0.05) \text{ m}$, calculate its tolerance.

Solution

Tolerance = upper limit – lower limit

$$\begin{aligned} \text{Tolerance} &= (6.3 + 0.05) \text{ m} - (6.3 - 0.05) \text{ m} \\ &= 6.35 \text{ m} - 6.25 \text{ m} \\ &= \mathbf{0.1 \text{ m}} \end{aligned}$$

Now that we have provided three examples on how to calculate tolerance, we want you to do activity 1 below. The activity you are about to do is similar to the example we have just discussed. This activity will help you to understand how to find greatest and least acceptable measurement using tolerance notation.



Activity 8

Find the greatest and least acceptable measurement for (5 ± 2) kg. Write down your answer and work in the space provided below.



Feedback

Feedback

After the examples we provided, you may have found the activity relatively easy to do. Your answer should have been similar to the one below.

The greatest and least acceptable measurements for (5 ± 2) kg are:

$$\text{Greatest measure} = 5 + 2 = 7 \text{ kg}$$

$$\text{Least measure} = 5 - 2 = 3 \text{ kg}$$

The activity you are about to do is similar to the one you have just done, but this time you have been given measurements of different units.



Activity 9

Write the range of measurements of 8 mm to 1.2 cm in tolerance notation. Write down your answer in the space provided below.



Feedback

Feedback

First we have to change 1.2 cm to millimetres by multiplying it by 10, since 1 cm = 10 mm.

Therefore, $1.2 \times 10 \text{ mm} = 12 \text{ mm}$

$$\begin{aligned} \text{Tolerance} &= 1.2 \text{ cm} - 8 \text{ mm} \\ &= 12 \text{ mm} - 8 \text{ mm} \\ &= 4 \text{ mm} \end{aligned}$$

$$\text{Absolute error} = \frac{4}{2} = 2 \text{ mm}$$

$$\begin{aligned} \text{True measure} &= \text{greatest} - \text{absolute error.} \\ &= 12 \text{ mm} - 2 \text{ mm} \\ &= 10 \text{ mm} \end{aligned}$$

Therefore, tolerance = $(10 \pm 2) \text{ mm}$

How did you fare with both Activities 1 and 2? Remember that you should always revise those sections that you did not get correct. If you got everything correct, congratulations!

Now let us look at how to find the sum of measurements.

Sum of Measurements

To find the sum of measurements, we add the upper limits of the length to get the maximum lengths and we add the lower limits of the length to get the minimum lengths.

Example 4: Consider the measurements 4.2 m and 1.8 m. Their sum can be worked out as follows:

$$\text{Least unit of measure} = 0.1 \text{ m}$$

$$\text{Absolute error} = 0.05 \text{ m}$$

$$\text{The range for first length} = (4.2 \pm 0.05) \text{ m}$$

$$\text{Upper limits} = 4.2 + 0.05 = 4.25 \text{ m}$$

$$\text{Lower limits} = 4.2 - 0.05 = 4.15 \text{ m}$$

$$\text{The range for second length} = (1.8 \pm 0.05) \text{ m.}$$

$$\text{Upper limits} = 1.8 + 0.05 = 1.85 \text{ m}$$

$$\text{Lower limits} = 1.8 - 0.05 = 1.75 \text{ m}$$

$$\text{Maximum sum} = (4.25 + 1.85) \text{ m} = \mathbf{6.10 \text{ m}}$$

$$\text{Minimum sum} = (4.15 + 1.75) \text{ m} = \mathbf{5.90 \text{ m}}$$

We can now look at how to calculate the difference of measurements.

Difference of Measurements

Like the way we find the sum of measurements, to find the difference of measurements we subtract the upper limits of the length to get the maximum lengths and we subtract the lower limits of the length to get the minimum lengths.

Let us do the following example together.

Example 5: Find the difference for the measurements 2.7 kg and 1.4 kg.

Solution

$$\text{Least unit of measurement} = 0.1 \text{ kg}$$

$$\text{Absolute error} = \frac{0.1}{2} = 0.05 \text{ kg}$$

$$2.7 \text{ kg lies between} = 2.7 \pm 0.05 \text{ kg}$$

$$\text{Upper limit} = 2.7 + 0.05 = \mathbf{2.75 \text{ kg}}$$

$$\text{Lower limit} = 2.7 - 0.05 = 2.65 \text{ kg}$$

$$1.4 \text{ kg lies between} = (1.4 \pm 0.05) \text{ kg}$$

$$\text{Upper limit} = 1.4 + 0.05 = \mathbf{1.45 \text{ kg}}$$

$$\text{Lower limit} = 1.4 - 0.05 = \mathbf{1.35 \text{ kg}}$$

$$\text{Maximum difference} = 2.75 - 1.35 = \mathbf{1.4 \text{ kg}}$$

$$\text{Minimum difference} = 2.65 - 1.45 = \mathbf{1.2 \text{ kg}}$$

Now you should apply what you have just learned by doing the next activity. You should be able to do the activity with minimum difficulty because it is similar to the example you have just studied. This activity will help you to understand how to find maximum and minimum differences.



Activity 10

A metal rod is 28 cm long. If a length of 20 cm is cut off, what are the limits of the remaining length? Write down your answer and working in the space below.



Feedback

Feedback

How was it? You would have realised that the limits of the remaining length can be worked as follows:

Least unit for both measurements = 1 cm

$$\text{Absolute error} = \frac{1 \text{ cm}}{2} = 0.5 \text{ cm}$$

28 cm lies between (28.5 ± 0.5) cm

$$\text{Upper limit} = 28 + 0.5 = 28.5 \text{ cm}$$

$$\text{Lower limit} = 28 - 0.5 = 27.5 \text{ cm}$$

20 cm lies between (20 ± 0.5) cm

$$\text{Upper limit} = 20 + 0.5 = 20.5 \text{ cm}$$

$$\text{Lower limit} = 20 - 0.5 = 19.5 \text{ cm}$$

$$\text{Maximum difference} = 28.5 - 19.5 = \mathbf{9.0 \text{ cm}}$$

$$\text{Minimum difference} = 27.5 - 20.5 = \mathbf{7.0 \text{ cm}}$$

Now let us discuss another example on maximum and minimum difference. The following example is similar to the activity you have just done. The only difference this time is that we will calculate the absolute.

Example 6: Find the absolute error of the difference of 8.6 km and 4.2 km.

The least unit of measurement = 0.1 km

So the absolute error = 0.05 km

First distance the limits is

Upper limits = $8.6 + 0.05 = 8.65$ km

Lower limits = $8.6 - 0.05 = 8.55$ km.

Second distance the limits is

Upper limits = $4.2 + 0.05 = 4.25$ km.

Lower limits = $4.2 - 0.05 = 4.15$ km

The maximum distance = $8.65 \text{ km} + 4.25 \text{ km}$
 $= 12.90 \text{ km}.$

The minimum distance = $8.55 \text{ km} + 4.15 \text{ km}$
 $= 12.70 \text{ km}$

$$\begin{aligned} \text{Absolute error} &= \frac{\text{maximum distance} - \text{minimum distance}}{2} \\ &= \frac{12.90 \text{ km} - 12.70 \text{ km}}{2} \\ &= \frac{0.20 \text{ km}}{2} \end{aligned}$$

Absolute error = **0.10 km**

We can also find the maximum and minimum of measures using products of measurement.

Product of Measurements

To find the product of measurements, we multiply the upper limits to get the maximum product and multiply the lower limits to get the minimum product.

Go through the following example carefully.

Example 7: Find the limits between which the area of 4 cm square lies.

Solution

Least unit of measurement = 1 cm

$$\text{Absolute error} = \frac{1 \text{ cm}}{2} = 0.5 \text{ cm}$$

$$\text{Upper limit} = 4 + 0.5 = 4.5 \text{ cm}$$

$$\text{Lower limit} = 4 - 0.5 = 3.5 \text{ cm}$$

Since the area of a square = $l \times l$, then

$$\text{Maximum area} = 4.5 \text{ cm} \times 4.5 \text{ cm} = 20.25 \text{ cm}^2$$

$$\text{Minimum area} = 3.5 \text{ cm} \times 3.5 \text{ cm} = 12.25 \text{ cm}^2$$

Therefore, the limits the area must lie between **20.25 cm²** and **12.25cm²**

Let us look at another example. The example is similar to the one you have just studied, but this time we are considering a rectangle.

Example 8: Find the limits between which the area of a rectangle with 8 cm length and 5 cm width lie.

Solution

Least unit for measurement = 1 cm

$$\text{Absolute error} = \frac{1 \text{ cm}}{2} = 0.5 \text{ cm.}$$

$$\text{Upper limit for the length} = 8 + 0.5 = 8.5 \text{ cm}$$

$$\text{Lower limit for the length} = 8 - 0.5 = 7.5 \text{ cm}$$

$$\text{Upper limit for width} = 5 + 0.5 = 5.5 \text{ cm}$$

$$\text{Lower limit for width} = 5 - 0.5 = 4.5 \text{ cm}$$

Since the area of a rectangle = $l \times b$, then

$$\text{Maximum area} = \text{upper limit length} \times \text{upper limit width}$$

$$= 8.5 \text{ cm} \times 5.5 \text{ cm}$$

$$= 46.75 \text{ cm}^2$$

$$\text{Minimum area} = \text{lower limit length} \times \text{lower limit width.}$$

$$= 7.5 \text{ cm} \times 4.5 \text{ cm}$$

$$= 33.75 \text{ cm}^2$$

Therefore, the limits for the area of a rectangle will be **46.75 cm²** and **33.75cm²**

Lastly, let us look at how to divide measurements.

Quotient of Measurement

We will focus on how to calculate the maximum mean and minimum mean. Go through the following example carefully.

Example 9: Find the maximum and minimum mean using the measures 5.3 kg, 6.5 kg and 9.9 kg, giving the answer to 3 significant figures.

Solution

First find the least unit of measure = 0.1 kg

$$\text{Absolute error} = \frac{0.1 \text{ kg}}{2} = 0.05 \text{ kg}$$

$$\begin{aligned} \text{Upper limits} &= (5.3 + 0.05) \text{ kg}, (6.5 + 0.05) \text{ kg}, (9.9 + 0.05) \text{ kg} \\ &= 5.35 \text{ kg}, 6.55 \text{ kg}, 9.95 \text{ kg} \end{aligned}$$

$$\begin{aligned} \text{Lower limits} &= (5.3 - 0.05) \text{ kg}, (6.5 - 0.05) \text{ kg}, (9.9 - 0.05) \text{ kg} \\ &= 5.25 \text{ kg}, 6.45 \text{ kg}, 9.85 \text{ kg} \end{aligned}$$

So

$$\text{Maximum mean} = \frac{\text{sum of all upper limits}}{\text{number of measures}}$$

$$= \frac{5.35 + 6.55 + 9.95}{3}$$

$$= \frac{21.85}{3}$$

$$= 7.283 \text{ kg}$$

$$= \mathbf{7.28 \text{ kg}}$$

$$\text{Minimum mean} = \frac{\text{sum of all lower limits}}{\text{number of measures}}$$

$$= \frac{5.25 + 6.45 + 9.85}{3}$$

$$= \frac{21.35}{3}$$

$$= 7.1833 \text{ kg}$$

$$= \mathbf{7.18 \text{ kg}}$$

Let us go through another example carefully.

Example 10: A piece of wire is 38 m long. If the wire is cut into five equal pieces, what will be the limits of the length of each piece?

Solution

The least unit of measure = 1 m.

$$\text{Absolute error} = \frac{1 \text{ m}}{2} = 0.5 \text{ m}$$

$$\text{Upper limit} = 38 + 0.5 = 38.5 \text{ m}$$

$$\text{Lower limit} = 38 - 0.5 = 37.5 \text{ m}$$

$$\text{Upper limit of the length of each piece} = \frac{38.5 \text{ m}}{5} = 7.7 \text{ m}$$

$$\text{Lower limit of the length of each piece} = \frac{37.5 \text{ m}}{5} = 7.5 \text{ m}$$

Therefore, the limits for the length for each piece of wire will be

7.7 m and 7.5 m

Topic 3: Summary



Summary

In this last topic of unit 1: Approximations and Scientific Notation, you learned how to calculate estimates of error and use approximations to estimate an error. You then learned how to calculate absolute error and relative error and use them to find the greatest and smallest acceptable measurements. Lastly you learned how to express the acceptable measures in tolerance notation and use it to find sum, difference, area and mean of the measurements.

You were given various examples of how to calculate estimates error and use approximations of error. Activities in the topic provided an opportunity for applying the concepts taught and you were encouraged to answer all the questions in the activity. If you heeded this request you should have been able to assess how well you understood the content of the topic and where necessary would have revised the sections that you had not understood at first attempt. If you did not do very well in the activity, this means that you needed to go over the material again.

You should now complete the Topic 3 Exercise to assess yourself on how well you have understood the content this Topic. This is last topic exercise in this unit. You are encouraged to answer all the questions.

After checking your answers and revising the sections where you may have had difficulties, you should read the Unit summary to remind you on what you learned in the whole unit. The references at the end of the Unit Summary are meant to encourage you to read widely on the subject of this unit.

Unit 1: Summary



Summary

In this unit there were three topics. In topic 1 you learned to approximate quantities and measures to a given degree of accuracy. This included how to round off numbers to the nearest unit and fraction of a unit. You further learned how to write numbers correct to the required number of decimal places and correct to the required number of significant figures.

In topic 2, you learned how to express numbers in scientific notation which is also called standard form.

In the third topic, you learned how to calculate estimates of error and use approximations to estimate an error.

By now you would have also completed the topic 3 exercise. This means that, besides the activities that you did within the topics, you have assessed your progress at the end of each of the three Topics.

Congratulations on completing the first unit of the grade 11 mathematics course. There is no tutor-marked assignment in this unit. We trust that the activities and topic exercises have adequately helped you to assess your own progress. You will be asked to complete and submit a tutor-marked assignment at the end of unit 3.

The next unit, unit 2, is on relations and functions. You will learn how to relate members of two or more sets according to a given rule and determine the relations that are functions.

References

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- Mukuyamba A. Et al 1995. *Mathematics Grade 11 Pupil's Book, Zambian Secondary School Syllabus*, Chingola: Yorvik Publishing Limited.
- Rayner D. 1988. *General Mathematics*, New York: Oxford University Press.

Topic Exercise 1



Exercise

Answer the following questions on a separate answer sheet.

1. Round off the following numbers to the nearest stated unit or fraction of the unit. (Each question carries 1 mark).
 - (a) The length 88.8 cm to the nearest cm.
 - (b) The mass of 19.1kg to the nearest kg.
 - (c) The potential difference 97.05 V to the nearest volt.
 - (d) The volume of milk 293.34 litres to the nearest tenth of a litre.
 - (e) A piece of wire 16.027 m to the nearest hundredth of a metre.

2. Round off 263.815 correct to
 - (a) the nearest hundredth
 - (b) the nearest ten
 - (c) one significant figure

3. A piece of wire 17.02 cm long is cut off from a wire of length 33.4 cm, what is the length of the remaining piece of wire to the nearest tenth of a metre?

4. Express the following numbers to the indicated number of decimal places. (Each question carries 1 mark).
 - (a) The number 16.005 to 2 decimal places
 - (b) The number 0.3536 to 3 decimal places

5. Work out 1.5×0.13 and leave your answer to 1 decimal place.

6. How many significant figures are there in the following numbers: (Each question carries 1 mark).
 - (a) 0.789
 - (b) 8.009
 - (c) 0.00100

7. Round off the following numbers to 3 significant figures. (Each question carries 1 mark).
- (a) 0.2375
 - (b) 18 918
8. Work out the following and leave the answer to 2 significant figures. (Each question carries 2 marks).
- (a) 0.15×1.5
 - (b) 6.72×600
9. Find the volume of water contained in a rectangular container of length 8.134m, breadth 6.154m and height 2.23m, giving your answers in litres correct to 2 significant figures.
10. A rectangular piece of land measures 116.66 m by 26.25m. Calculate the area of the land and give the answer to three significant figures.

Now compare your answers with those provided at the end of this unit.

Topic Exercise 2



Exercise

- Express the following numbers in standard form.
 - 3 001
 - 900 000
 - 0.070 2
 - 523.4
 - 0.000 068
- The area of the Indian Ocean is about 73 000 000 km². Write this in standard form.
- The width of a HIV virus is 0.000 000 028m. Write this in standard form.
- Work out the following giving your answer in standard form.
 - $1\,500 \times 8\,000\,000$
 - $8\,000 \div 0.004$
- Express the following in standard form.
 - The atomic weight of an electron, 0.0005488 g
 - The mass of 1 cm³ of hydrogen, 0.0008989 g
- Express 0.0345 in scientific notation correct to 2 significant Figures
- The number of orphans and vulnerable in one province of Zambia is 599 900.
 - Write this number in standard form.
 - Express this number correct to 1 significant figure.

Now compare your answers with those provided at the end of this unit.

Topic Exercise 3



Exercise

- Find the absolute error of the following :
 - 182.345 l
 - 65.855 km
- Find the upper limit and lower limit of 0.008 m.
- In each of the following, find the relative error and percentage error to 3 significant figures.
 - 0.34 km.
 - 3.0 cm (correct to 2 significant figures)
- In the following give the greatest and smallest which are acceptable:
 - (4 ± 0.5) m
 - (0.61 ± 0.02) cm
- In each of the following find the tolerance, given that the acceptable measures lie between:
 - 8.55 kg and 8.40 kg
 - 4.93 cm^2 and 5.00 cm^2
- Write the following in tolerance notation.
 - 79 m to 83 m
 - 4.625 kg to 4.633 kg
- Find the maximum and the minimum sums of the following measurements.
 - 7 m and 11 m
 - 16.2 ml and 19.3 ml
- The sides of a triangle are said to have length 4.5 cm, 3.9 cm and 3.4 cm, find the limits between which the perimeter must lie.
- In The following find the upper and lower limits of the true difference of the measurements given.
 - 4.3 cm and 7.5 cm
 - 1.42 m and 0.90 m
- A metal strip 9.8 cm long is cut from a piece of length 50.0 cm. If each length has a tolerance of 0.2 cm, what are the

limits of the remaining piece?

11. Find the limit between the area of a rectangle with length 9.4 cm and breadth 2.4 cm, to the nearest 0.2 cm must line.
12. A charitable organisation contributed blankets to HIV/AIDS patients of length 650 m, 660 m, 500m and 570 m. What are the limits of the mean length?
13. A sum of money amounting to K(300 000 \pm 6000) is shared among 10 people . What are the limits of the answers each person gets?

Now compare your answers with those provided at the end of this unit.

Answers for Unit 15 Topic Exercises

Answers to Topic Exercise 1



Answers to Assignment

1. (a) $88.8 \text{ cm} = \mathbf{89 \text{ cm}}$, to the nearest centimetre.
 (b) $19.1 \text{ kg} = \mathbf{19 \text{ kg}}$, the nearest kilogram
 (c) $97.05 \text{ v} = \mathbf{97 \text{ v}}$, to the nearest volt
 (d) $273.34 \text{ litres} = \mathbf{273.3 \text{ litres}}$ to the nearest tenth of a litre
 (e) $16.027 \text{ m} = \mathbf{16.03 \text{ m}}$ to the nearest hundredth of a metre.

2. (a) $263.815 = \mathbf{263.82}$ correct to the nearest hundredth
 (b) $263.815 = \mathbf{260}$ correct to the nearest ten
 (c) $263.815 = \mathbf{300}$ correct to one significant figure

3. $33.4 \text{ cm} - 17.02 \text{ cm} = 16.38 \text{ cm}$
 $= \mathbf{16.4 \text{ cm}}$ to the nearest tenth of a metre

4. (a) $16.005 = \mathbf{16.01}$ to 2 decimal places
 (b) $0.3536 = \mathbf{0.354}$ to 3 decimal places

5. $1.5 \times 0.13 = 0.195$
 $= 0.2$ to 1 decimal place.

6. (a) $0.789 =$ there are 3 significant figures
 (b) $8.009 =$ there are 4 significant figures
 (c) $0.00100 =$ there are 3 significant figures

7. (a) $0.2375 = \mathbf{238}$ to 3 significant figures
 (b) $18\,918 = \mathbf{18900}$ to 3 significant figures

8. (a) $0.15 \times 1.5 = 0.225$
 $= \mathbf{0.23}$ to 2 significant figures
 (b) $6.72 \times 600 = 4032$
 $= \mathbf{4000}$ to 2 significant figures

9. Volume = length \times breadth \times height of water contained in a

$$\begin{aligned} &= 8.134\text{m} \times 6.154\text{m} \times 2.23\text{m} \\ &= 111.62629828 \text{ m}^3 \\ &= \mathbf{110 \text{ m}^3} \text{ to 2 significant figures} \end{aligned}$$

10. Area = length \times width

$$\begin{aligned} &= 116.66 \text{ m} \times 26.25 \text{ m} \\ &= 3062.325 \text{ m}^2 \\ &= \mathbf{3060 \text{ m}^2} \text{ to 3 significant figures} \end{aligned}$$

We hope that after comparing your answers with the model answers provided in the above feedback you might have got all the answers correct. Congratulations! You are ready to move on to the next topic. If not then we suggest you try the questions you got wrong again. You may be required to read through the section again in order to gain more understanding

Answers to Topic Exercise 2



Answers to Assignment

1. (a) $3\ 001 = 3.001 \times 10^3$
 (b) $900\ 000 = 9 \times 10^5$
 (c) $0.070\ 2 = 7.02 \times 10^{-2}$
 (d) $523.4 = 5.234 \times 10^2$
 (e) $0.000\ 068 = 6.8 \times 10^{-5}$
2. $73\ 000\ 000\ \text{km}^2 = 7.3 \times 10^7\ \text{km}^2$
3. $0.000\ 000\ 028\ \text{m} = 2.8 \times 10^{-8}\ \text{m}$
4. (a) $1\ 500 \times 8\ 000\ 000 = (1.5 \times 10^3) \times (8 \times 10^6)$
 $= (1.5 \times 8) \times (10^3 \times 10^6)$
 $= 12 \times 10^{3+6}$
 $= 12 \times 10^9$
 $= \mathbf{1.2 \times 10^{10}}$
 (b) $8\ 000 \div 0.004 = 8 \times 10^3 \div 4 \times 10^{-3}$
 $= (8 \div 4) \times (10^3 \div 10^{-3})$
 $= 2 \times 10^{3-(-3)}$
 $= 2 \times 10^{3+3}$
 $= \mathbf{2 \times 10^6}$
5. (a) The atomic weight of an electron
 $= 0.0005488\ \text{g}$
 $= \mathbf{5.488 \times 10^{-4}\ \text{g}}$
 (b) The mass of $1\ \text{cm}^3$ of hydrogen
 $= 0.0008989\ \text{g}$
 $= \mathbf{8.989 \times 10^{-4}\ \text{g}}$
6. $0.0345 = 3.45 \times 10^{-2}$
 $= \mathbf{3.5 \times 10^{-2}}$ correct to 2 significant figures.
7. (a) $599\ 900 = 5.999 \times 10^5$
 (b) $599\ 900 = 600\ 000$ correct to 1 significant figure.

We hope that after comparing your answers with the model answers and work provided in the above feedback you might have

got all the answers correct. Congratulations! If not then we suggest you revise and rework the questions you got wrong again. You may be required to read through the section again in order to gain more understanding.

Answers to Topic Exercise 3



Answers to Assignment

1. (a) The least unit of measurement of 182.345 l is 0.001 l

$$\text{Absolute error} = \frac{1}{2} \times 0.001 \text{ l} = \mathbf{0.0005 \text{ l}}$$

- (a) The least unit of measurement of 65.855 km is 0.001 km

$$\text{Absolute error} = \frac{1}{2} \times 0.001 \text{ km} = \mathbf{0.0005 \text{ km}}$$

2. The least unit of measurement of 0.008 m is 0.001 m.

$$\text{Absolute error} = \frac{0.001 \text{ m}}{2} = 0.0005 \text{ m}$$

$$\begin{aligned} \text{Upper limit} &= 0.008 + 0.0005 \\ &= \mathbf{0.0085 \text{ m}} \end{aligned}$$

$$\begin{aligned} \text{Lower limit} &= 0.008 - 0.0005 \\ &= \mathbf{0.0075 \text{ m}} \end{aligned}$$

3. (a) The least unit of measurement of 0.34 km is 0.01 km

$$\text{Absolute error} = \frac{0.01 \text{ km}}{2} = 0.005 \text{ km}$$

$$\begin{aligned} \text{Relative error} &= \frac{0.005 \text{ km}}{0.34 \text{ km}} \\ &= \frac{0.005 \text{ km} \times 1000}{0.34 \text{ km} \times 1000} \\ &= \frac{5}{340} \\ &= \frac{1}{68} \end{aligned}$$

$$\text{Percentage error} = \frac{1}{68} \times 100\%$$

$$= \mathbf{1.47\%} \text{ (correct to 3 significant figures)}$$

- (b) The least unit of measurement of 3.0 cm is 0.1 cm

$$\text{Absolute error} = \frac{0.1 \text{ cm}}{2} = 0.05 \text{ cm}$$

$$\begin{aligned} \text{Relative error} &= \frac{0.05 \text{ cm}}{3.0 \text{ cm}} \\ &= \frac{0.05 \text{ cm} \times 100}{3.0 \text{ cm} \times 100} \\ &= \frac{5}{300} \\ &= \frac{1}{60} \end{aligned}$$

$$\text{Percentage error} = \frac{1}{60} \times 100\%$$

$$= \mathbf{1.67\%} \text{ (correct to 3 significant figures)}$$

4. (a) $(4 \pm 0.5) \text{ m}$

$$\text{The greatest measure} = (4 \pm 0.5) \text{ m} = \mathbf{4.5 \text{ m}}$$

$$\text{The smallest measure} = (4 - 0.5) \text{ m} = \mathbf{3.5 \text{ m}}$$

- (b) $(0.61 \pm 0.02) \text{ cm}$

$$\text{The greatest measure} = (0.61 + 0.02) \text{ cm} = \mathbf{0.63 \text{ cm}}$$

$$\text{The smallest measure} = (0.61 - 0.02) \text{ cm} = \mathbf{0.59 \text{ cm}}$$

5. (a) 8.55 kg and 8.40 kg

$$\text{Tolerance} = (8.55 - 8.40) \text{ kg} = \mathbf{0.15 \text{ kg}}$$

- (b) 4.93 cm^2 and 5.00 cm^2

$$\text{Tolerance} = (5.00 - 4.93) \text{ cm}^2 = \mathbf{0.07 \text{ cm}^2}$$

6. (a) 79 m to 83 m

$$\text{Tolerance} = (83 - 79) \text{ m} = 4 \text{ m}$$

$$\text{Absolute error} = \frac{4 \text{ m}}{2} = 2 \text{ m}$$

$$\begin{aligned} \text{True measure} &= \text{greatest measure} - \text{absolute error}, \\ &= 83 \text{ m} - 2 \text{ m} \\ &= 81 \text{ m} \end{aligned}$$

$$\text{Therefore, tolerance} = \mathbf{(81 \pm 2) \text{ m}}$$

- (b) 4.625 kg to 4.633 kg

$$\text{Tolerance} = (4.633 - 4.625) \text{ kg} = 0.008 \text{ kg}$$

$$\text{Absolute error} = \frac{0.008 \text{ kg}}{2} = \mathbf{0.004 \text{ kg}}$$

$$\begin{aligned} \text{True measure} &= \text{greatest measure} - \text{absolute error} \\ &= 4.633 - 0.004 \\ &= 4.629 \text{ kg} \end{aligned}$$

$$\text{Therefore, tolerance} = \mathbf{(4.629 \pm 0.004) \text{ kg}}$$

7. (a) 7 m and 11 m

$$\text{The least unit for both measurements} = 1 \text{ m}$$

$$\text{Absolute error} = \frac{1 \text{ m}}{5} = 0.5 \text{ m}$$

$$\text{First length} = (7 \pm 0.5) \text{ m}$$

$$\text{Upper limit} = 7 + 0.5 = 7.5 \text{ m}$$

$$\text{Lower limit} = 7 - 0.5 = 6.5 \text{ m}$$

$$\text{Second length} = (11 \pm 0.5) \text{ m}$$

$$\text{Upper limit} = 11 + 0.5 = 11.5 \text{ m}$$

$$\text{Lower limit} = 11 - 0.5 = 10.5 \text{ m}$$

$$\text{Maximum sum} = (7.5 + 11.5) \text{ m} = \mathbf{19 \text{ m}}$$

$$\text{Minimum sum} = (6.5 + 10.5) \text{ m} = \mathbf{17 \text{ m}}$$

(b) 16.2 ml and 19.3 ml

$$\text{The least unit for both measurements} = 0.1 \text{ ml}$$

$$\text{Absolute error} = \frac{0.1 \text{ ml}}{2} = 0.05 \text{ ml}$$

$$\text{First volume} = (16.2 \pm 0.05) \text{ ml}$$

$$\text{Upper limit} = 16.2 + 0.05 = 16.25 \text{ ml}$$

$$\text{Lower limit} = 16.2 - 0.05 = 16.15 \text{ ml}$$

$$\text{Second volume} = (19.3 \pm 0.05) \text{ ml}$$

$$\text{Upper limit} = 19.3 + 0.05 = 19.35 \text{ ml}$$

$$\text{Lower limit} = 19.3 - 0.05 = 19.25 \text{ ml}$$

$$\text{Maximum sum} = (19.35 + 16.25) \text{ ml} = \mathbf{35.6 \text{ ml}}$$

$$\text{Minimum sum} = (19.25 + 16.15) \text{ ml} = \mathbf{34.4 \text{ ml}}$$

8. The least unit for measurements of the triangle of sides 4.5 cm, 3.9 cm and 3.4 cm = 0.1 cm

$$\text{Absolute error} = \frac{0.1}{2} = 0.05 \text{ cm}$$

$$\text{Length of the first triangle} = (4.5 \pm 0.05) \text{ cm}$$

$$\text{Upper limit for 4.5 cm} = 4.5 + 0.05 = 4.55 \text{ cm}$$

$$\text{Lower limit for 4.5 cm} = 4.5 - 0.05 = 4.45 \text{ cm}$$

$$\text{Length of the second side of triangle} = (3.9 \pm 0.05) \text{ cm}$$

$$\text{Upper limit for 3.9 cm} = 3.9 + 0.05 = 3.95 \text{ cm}$$

$$\text{Lower limit for 3.9 cm} = 3.9 - 0.05 = 3.85 \text{ cm}$$

$$\text{Length of the third triangle} = (3.4 \pm 0.05) \text{ cm}$$

Upper limit for 3.4 cm = 3.4 cm + 0.05 = 3.45 cm

Lower limit for 3.4 cm = 3.4 – 0.05 = 3.35 cm

Maximum perimeter = 4.55 + 3.95 + 3.45 = **11.95 cm**

Minimum perimeter = 4.45 + 3.85 + 3.35 = **11.65 cm**

9. (a) 4.3 cm and 7.5 cm

Absolute error = $\frac{0.1}{2} = 0.05$ cm

4.3 cm lies between $(4.3 \pm 0.05) = 4.35$ cm

Upper limit = 4.3 + 0.05 = 4.35 cm

Lower limit = 4.3 – 0.05 = **4.25 cm**

7.5 cm lies between (7.5 ± 0.05) cm

Upper limit = 7.5 + 0.05 = 7.55 cm

Lower limit = 7.5 – 0.05 = 7.45 cm

Maximum difference = 7.55 – 4.25 = **3.30 cm**

Minimum difference = 7.45 – 4.35 = **3.10 cm**

(b) 1.42 m and 0.90 m

Absolute error = $\frac{0.01}{2} = 0.005$ m

1.42 m lies between (1.42 ± 0.005) m

Upper limit = 1.42 + 0.005 = **1.425 m**

Lower limit = 1.42 – 0.005 = 1.415 m

0.90 m lies between (0.90 ± 0.005) m

Upper limit = 0.90 + 0.005 = 0.905 m

Lower limit = 0.90 – 0.005 = **0.895 m**

Maximum difference = 1.425 – 0.895 = **0.530 m**

Minimum difference = 1.415 – 0.905 = **0.510 m**

10. The limits of the remaining metal strip after cutting 9.8 cm from a piece of length 50.0 cm is

$$\text{Tolerance} = 0.2 \text{ cm}$$

$$\text{Absolute error} = \frac{0.2 \text{ cm}}{2} = 0.1 \text{ cm}$$

$$9.8 \text{ cm lies between } (9.8 \pm 0.1) \text{ cm}$$

$$\text{Upper limits} = 9.8 + 0.1 = 9.9 \text{ cm}$$

$$\text{Lower limit} = 9.8 - 0.1 = 9.7 \text{ cm}$$

$$50.0 \text{ cm lies between } (50.0 \pm 0.1) \text{ cm}$$

$$\text{Upper limits} = 50.0 + 0.1 = 50.1 \text{ cm}$$

$$\text{Lower limit} = 50.0 - 0.1 = 49.9 \text{ cm}$$

$$\text{Maximum difference} = 49.9 - 9.7 = \mathbf{40.2 \text{ cm}}$$

$$\text{Minimum difference} = 50.1 - 9.7 = \mathbf{40.24 \text{ cm}}$$

11. The limit between the area of a rectangle with length 9.4 cm and breadth 2.4 cm, to the nearest 0.2 cm must line:

$$\text{Least unit of measurement} = 0.2 \text{ cm}$$

$$\text{Absolute error} = \frac{0.2}{2} = 0.1 \text{ cm}$$

$$\text{Limit of length of rectangle} = (9.4 \pm 0.1) \text{ cm}$$

$$\text{Upper limits} = (9.4 + 0.1) \text{ cm} = 9.5 \text{ cm}$$

$$\text{Lower limits} = 9.4 - 0.1 = 9.3 \text{ cm}$$

$$\text{Limit of breadth} = (2.4 \pm 0.1) \text{ cm}$$

$$\text{Upper limit} = 2.4 + 0.1 = 2.5 \text{ cm}$$

$$\text{Lower limit} = 2.4 - 0.1 = 2.3 \text{ cm.}$$

$$\text{Maximum area} = 9.5 \times 2.5 = \mathbf{23.75 \text{ cm}^2}$$

$$\text{Minimum area} = 9.3 \times 2.3 = \mathbf{21.39 \text{ cm}^2}$$

12. The limits of the mean length of lengths 650 m, 660m, 500m and 570m are:

$$\text{Least unit of measurement} = 1 \text{ m}$$

$$\text{Absolute error} = \frac{1}{2} = 0.5 \text{ m}$$

$$\begin{aligned} \text{Upper limits} &= (650 + 0.5) \text{ m}, (660 + 0.5) \text{ m}, \\ &\quad (500 + 0.5) \text{ m and } (570 + 0.5) \text{ m} \\ &= 650.5 \text{ m}, 660.5 \text{ m}, 500.5 \text{ m and } 570.5 \text{ m} \end{aligned}$$

$$\begin{aligned} \text{Lower limits} &= (650 - 0.5) \text{ m}, (660 - 0.5) \text{ m}, \\ &\quad (500 - 0.5) \text{ m, and } (570 - 0.5) \text{ m} \\ &= 649.5 \text{ m}, 659.5 \text{ m}, 499.5 \text{ m and } 569.5 \text{ m} \end{aligned}$$

$$\begin{aligned} \text{Maximum mean} &= \frac{650.5 + 660.5 + 500.5 + 570.5}{4} \\ &= \frac{2382}{4} \\ &= \mathbf{595.50 \text{ m}} \end{aligned}$$

$$\begin{aligned} \text{Minimum mean} &= \frac{649.5 + 659.5 + 499.5 + 569.5}{4} \\ &= \frac{2378}{4} \\ &= \mathbf{594.50 \text{ m}} \end{aligned}$$

13. The limits of the sum of money each person gets are:

$$\text{Upper limit} = \text{K}300\,000 + \text{K}6000 = \text{K}306\,000$$

$$\text{Lower limit} = \text{K}300\,000 - \text{K}6000 = \text{K}294\,000$$

$$\text{Maximum mean} = \frac{\text{K}306\,000}{10} = \mathbf{\text{K}30\,600}$$

$$\text{Minimum mean} = \frac{0294000}{10} \frac{\text{K}294\,000}{10} = \mathbf{\text{K}29\,400}$$

We hope that after comparing your answers with the model answers and work provided in the above feedback you might have got all the answers correct. Congratulations! If not then we suggest you try the questions you got wrong again. You may be required to read through the section again in order to gain more understanding.

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Unit 2

Relations and Functions

Introduction

Welcome to the second unit in your Mathematics Grade 11 course. You have just completed Unit 1: Approximations and Scientific notations in which you learnt about Approximations and Scientific notation. You learnt how to round off numbers to a given degree of accuracy. You learnt how to write numbers correct to the required number of decimal places and correct to the required number of significant figures. You further learnt how to express numbers in scientific notation and to calculate estimates of error and use approximations to estimate an error.

This unit is about Relations and Functions. The study on Relations and Functions is important simply because in life many things are related to one another in various ways. For example, among human beings, we have relations that exist between the parents and their children, uncles or aunts and their nephews and nieces and so forth. We also have other types of relations that exist between moving objects and time taken such as the speed of a moving car and the time taken to travel a given distance. In business we have relations that exist between the money invested in business and the profit made or the interest gained on money deposited in the bank and the time taken at a given rates. These are some of the examples on how the topic of relations and functions apply in everyday life.

This unit has 2 topics. As you study through them, you will come across some activities and feedback will be provided thereafter. These activities are meant to give you some time to practice what you have learned so that you develop skills on problem solving. Soon after completing each topic you will be required to do a topic exercise placed at the end of the unit. You are encouraged to answer all the questions in the topic exercises and to check your work against the model answers only after completing the exercises.

This unit does not have a tutor - marked assignment. Therefore, you will assess your performance by doing Topic Exercises 1 and 2.

Upon completion of this unit you will be able to:



Outcomes

- *Relate* members of two or more sets according to a given rule.
- *Determine* relationships that are functions.
- *Use* function notation.
- *Find* inverse of a function.

Timeframe



How long?

We estimate that to complete this unit you will need between 8 and 11 hours. This time includes the time you will spend in doing the activities checking them against the feedback. If you do not finish studying the unit within this estimated time do not worry since we do not learn at the same pace.

You are encouraged to spend about 1 hour answering the first topic exercise and two hours on the second topic exercise in this unit. Since there are two topic exercises in this unit you are expected to spend 3 hours on them.

The total hours for completing the unit will thus be between 11 and 14 hours.

Learning Resources



Resources

In order to study the Relations and Functions with minimal difficulties you will need the following materials:

- A ruler
- A pencil or a pen
- A calculators
- Graph papers

Teaching and Learning approaches

In this unit we have used three teaching and learning methods in presenting the content. These methods are:

- **Conceptual:** The unit of relations and functions is based on the basic knowledge of the sets which form facts, concepts, rules and generalisations or formulas. This method will help you understand the meaning of facts, rules, formulas and procedures used in Relations and Functions;
- **Problem-solving:** Relations and functions are about seeing how objects or things are related or compare with one another or which patterns these objects have in common. The method of problem solving therefore will help you solve mathematical problems that relate to real life situations by the use of procedures and concepts you shall develop as you go through the unit. You will also be able to discuss mathematical problems, answers and strategies with friends.
- **Skills:** Relations and Functions are presented with specific skills on how to solve problems. The skills are meant to be used in the critical analysis of the elements of the given sets in order to understand the relationships between these sets. These skills will help you on how to use the facts, rules, formulas as you do self marked activities and topic exercises in relations and functions.

As you read through the unit, do the activities and exercises and discuss your ideas with other learners and your tutor. This will make you to practice the three teaching and learning approaches and introduce you to other points of view.

The topic of Relations and Functions has some terms with specific meanings. You are required to study them but if they are not so clear do not worry as they would be explained in detail in the text. Some of these terms are explained below.



Terminology

Arrow Diagram:	A diagram with arrows joining element(s) of the domain to element(s) of the range.
Domain:	The first set in a relation. The set from which the relation draws the elements.
Function:	A relation in which all elements of the domain have images and each object has only one image in the range.
Image or output:	Element of the Range.
Independent variable:	A letter (variable) that can assume any numerical value without depending on another variable.
Mapping:	A relation in which every element of the domain is paired with only one element in the range.
Objective or input:	Element or member of the Domain.
Range:	The second set after the operation (relation). The set formed after the rule.
Relation:	A rule that pairs the elements of two sets.

(Source: John A. Fyfield and Dudley Blane (2000). *The School Mathematics Dictionary*)

Let us proceed with the first topic where we will discuss the relationships of members of two or more sets according to a given rule.

Topic 1: Relations

In this topic we will focus on relationships of members of two or more sets according to a given rule. You will learn how to find the domain and the range. You will further learn how to represent the relations using the arrow diagram, sets of ordered pairs, formula and Cartesian graphs. We will also discuss the different types of relations.

Like in all other topics of this course, after studying through this topic you will be required to work out Topic Exercise 1 and mark your own work using the feedback provided at the end of the unit. You are encouraged not to go through the feedback before doing the topic exercise.

In this first topic of the unit, we will address the first of the four unit outcomes, namely:

- Relate members of two sets according to a given rule.

We have divided this outcome into the following objectives:



Objectives

- *Find* the range.
- *Find* the domain.
- *Represent* the relations using the arrow diagram, sets of ordered pairs, formula and Cartesian graphs.
- *State* the types of relations.

The objectives you have just gone through are meant to help you gain an insight of what you ought to achieve at the end of the topic. Some of the ways in which we can help you gain knowledge and skills based on the objectives we have just outlined are through an introduction to relevant content and working through a number of the activities.

Let us proceed with our study by first understanding what relations are.

What do we mean by relations?

The ordinary meaning of the word “relate” is to compare. The mathematical usage is not very different but means a rule or defining statement about the similarities of the elements of two sets. You are going to learn how to use this rule to find the domain (the first set) and the range (the second set) and represent the relation in various ways will take you through them step by step.

Let us begin our study by discussing domain and the range.

Domain and Range

The meaning of both terms, the domain and the range, are given in the section of terminology in this unit. If you cannot remember the meanings correctly then you are encouraged to go back and read through them again. However, we will discuss their meanings again within the text as we go along with our study.

What do you understand about the domain and the range?

To answer this question you are encouraged to do the following the activity first.



Activity 1

Consider the following two sets and determine the relationship that may exist between them.

$A = \{ \text{Zambia, Namibia, Botswana} \}$

$B = \{ \text{Lusaka, Windhoek, Gaborone} \}$

Write your answer in the space below.



Feedback

Feedback

We hope you are quite familiar with the elements in both sets. You might have noticed that Set A is made up of the names of three countries in Africa and Set B is made up of their capital cities. That is

- Lusaka is the capital city of Zambia
- Windhoek is the capital city of Namibia
- Gaborone is the capital city of Botswana

You might have also noticed that the elements of the two sets have been matched such that they form pairs. This pairing is defined by a rule: “is the capital city of”. This rule is called **a relation** between the two sets.

From the example of the countries and their respective cities given above we see the elements of the countries forming one set and the elements of the cities forming another set.

The first set (set A) is called the **domain** and the second set (set B) is called the **range**. Do you know what the elements of the domain and the range are called? Of course, the elements of the domain are called **objects** and the elements of the range are called the **images**.

Now, how would you define a relation? Based on what we have just discussed, the **relation** can be defined as the rule or condition under which after being applied on the domain, results in formation of a set known as the range. Therefore, we can define a domain set; as a set where the rule or the relation will apply.



Note

You should note that, to define a relation, we require two sets and a rule which associates the elements of the two sets.

When we know the domain and the rule that associates the elements of two sets we can find the range. To illustrate this fact let us do the following example.

Example 1

Consider a relation that is defined with domain $\{4, 6, 8, 10\}$. Find the range if the relation is “is half of”.

Solution

To find the elements of the range we have to multiply each element in the domain by half. We show the workings as follows

The first element is 4, then $\frac{1}{2} \times 4 = 2$

The second element is 6, then $\frac{1}{2} \times 6 = 3$

The third element is 8, then $\frac{1}{2} \times 8 = 4$

The fourth element is 10, then $\frac{1}{2} \times 10 = 5$

The answers we have obtained in our calculations are 2, 3, 4 and 5. These numbers form the images of set B which is the range.

Therefore, the range is

$$B = \{2, 3, 4, 5\}$$

From the example we have just gone through, you have learnt how to find the range given the domain and the relation. You should also be aware that we could also be required to find the domain given the range and the rule that associates the elements of the two sets. In other words, this is the reverse of what you have just learnt. Therefore, you should use the knowledge you have gained to do the following activity.



Activity 2

A relation is defined with range $\{4, 5, \text{ and } 6\}$. Find the domain if the relation is “plus 3 is”.

Write down your answer in the space below.



Feedback

Feedback

You might have realised that in order to arrive at the images $\{4, 5, 6\}$, the rule “plus 3 is” was used. Therefore, to reverse the operation we need to subtract 3 from all the images of the range. That is;

For the first image , we have $4 - 3 = 1$

For the second image, we have $5 - 3 = 2$

For the last image, we have $6 - 3 = 3$

Therefore, the domain is $\{1, 2, 3\}$

So far you have learned how to find the range when the domain and the relation are given. You have also learnt to find the domain when the range and the relation are given.

Let us now look at different ways of representing the relations.

Representation of Relations

The relations can be either number relations or non-number relations. What do think is the difference between number relations and non-number relations?

Of course, number relations are relations that involve numbers only such as the one we discussed in Activity 2 above whereas, non-numbers relations are relations that do not involve numbers. One example of non-number relations is the one we discussed in Activity 1 in this topic.

There are **four** ways of representing number relations. These are arrow diagrams, sets of ordered pairs, formula and Cartesian graphs. On the other hand, there are **three** ways of representing non-number relations. These are arrow diagram, sets of ordered pairs and Cartesian graphs. You might have noticed that formula is not one of the ways of representing non-number relations. Therefore, we can say that non-number relations cannot be represented by formulae.

Now, let us first see how relations can be represented using the arrow diagrams.

The Arrow Diagram

The arrow diagram is one of the ways of representing relation between two sets. It comprises the domain and the range, with the arrows connecting objects to their images.

So we can define an arrow diagram as a diagram with two circular or rectangular sets and arrows joining elements between them.

Let us do the following example to illustrate this definition.

Example 2

Consider the relation “is a prime factor of” from $A = \{2, 3, 5\}$ to $B = \{4, 6, 8, 9, 10\}$. Represent this relation by an arrow diagram.

Solution

The figure below shows the relation A “is a factor of” B

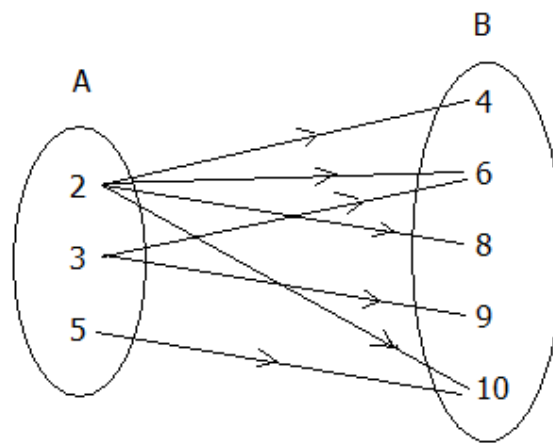


Figure 1: Arrow diagram for A “is a prime factor” B

From the arrow diagram, we see that

- The arrows from 2 are connected to 4, 6, 8 and 10. That is 2 “is the factor of” 4, 6, 8 and 10.
- The arrows from 3 are connected to 6 and 9. That is 3 “is the factor of” 6 and 9.
- The arrow from 5 is connected to 10.

Now you should do the following activity. It is similar to the example we have just gone through. It will help you understand better how to draw the arrow diagram.



Activity 3

Consider the relation “plus 4 is” from $C = \{1, 2, 3\}$ to $D = \{5, 6, 7\}$. Represent this relation by an arrow diagram.

Write down your answer in the space provided below.



Feedback

Feedback

In order to find the images of the range, we add 4 to each of the objects in the domain as follows.

$$1 + 4 = 5$$

That is 1 is connected to 5.

$$2 + 4 = 6$$

Thus, the number 2 in the domain set is connected to 6 in the range set.

$$3 + 4 = 7$$

Therefore, the number 3 is connected to 7 in the range set.

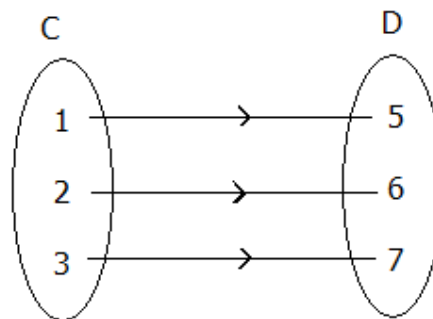


Figure 2: Arrow diagram for C “plus 4 is” D

So far we have looked at how to represent relations using the arrow diagrams. Now we will look at another method of representing the relations. This time we look at sets of ordered pairs.

Sets of Ordered Pairs

A set of ordered pairs can be formed from the objects and their images. The pairs are ordered in such a way that the first element of the pair comes from the domain and the second element comes from the range.

To illustrate this fact, let us look at the following example.

Example 3

Represent the following relation as a set of ordered pairs. The relation “is a prime factor of” from $A = \{2, 3, 5\}$ to $B = \{4, 6, 8, 9, 10\}$.

Solution

When we examine the two sets, the element 2 in set A has four images in set B which are 4, 6, 8 and 10. The object and its images can be written as ordered pairs as

2 is the factor of 4, as an ordered pair it is written as (2, 4)

2 is the factor of 6, as an ordered pair it is written as (2, 6)

2 is the factor of 8, as an ordered pair it is written as (2, 8)

2 is the factor of 10, as an ordered pair it is written as (2, 10).

Let us move to the next element in set A. The next number is 3. It has two images in set B. These images are 6 and 9. Therefore, the ordered pairs of the number 3 and its two images are (3, 6) and (3, 9).

The last element in set A is 5. It has one image in set B. It can be written as an ordered pair as (5, 10).

Therefore, the relation “is a prime factor of” from $A = \{2, 3, 5\}$ to $B = \{4, 6, 8, 10\}$ can be written as a set of ordered pairs as $\{(2, 4), (2, 6), (2, 8), (2, 10), (3, 6), (3, 9), (5, 10)\}$

Now you should practice what you have just learnt by doing the activity that follows.

**Activity 4**

Consider the relation “plus 4 is” from $C = \{1, 2, 3\}$ to $D = \{5, 6, 7\}$. Represent this relation as a set of ordered pairs

Write down your answer in the space provided below.



Feedback

Feedback

When we add 4 to each object in domain we have

$$1 + 4 = 5$$

The ordered pair is (1,5)

$$2 + 4 = 6$$

The ordered pair is (2, 6)

$$3 + 4 = 7$$

The ordered pair is (3, 7)

Therefore, the set of ordered pair is $\{(1, 5), (2, 6), (3, 7)\}$

So far we have looked at two ways of representing the relations. These are arrow diagrams and sets of ordered pairs. Now let us look at the third method. This is a formula.

Formula

We can define a formula as an equation giving the relation between two or more quantities.

If $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$, what relation do you find that exist between them? Of course the relation between set A and set B is “plus 3 is”.

If x represents a member of the domain (that is set A) and y represents a member of the range (that is set B), then the relation is

‘ x plus 3 is y ’.

This may be expressed as

y is equal to x plus 3’,

which gives the formula as:

$$y = x + 3.$$

We may also be required to write the relation in a set builder notation.

Set builder notation is one of the three ways you learnt in describing a set in Grade 10. Set builder notation is also referred to as rule method. That is, a rule is used to describe a set.

When presenting a set using a set builder notation, we use a letter to represent elements of the set. For example, if x represents elements in set A which is a set of even numbers greater than 10 but less than 20, in set builder notation this can be written as:

$$A = \{X: 10 < x < 20, x \in \text{Even numbers}\}$$

The other two ways of describing the set is by listing the elements and by describing the set in words.

To represent the relation of set A and set B, we need first to express the sets as ordered pairs.

Using the formula $y = x + 3$, the set $R = \{(1, 4), (2, 5), (3, 6)\}$ of the ordered pairs we can repressed it in set builder notation as:

$R = \{(x, y): y = x + 3, x \in A, y \in B\}$. This is read as “R is the set of all ordered pairs (x, y), such that $y = x + 3$, where $x \in A$ and $y \in B$.”

Let us do the following example.

Example 4

Given that $X = \{2, 4, 6, 8\}$ and $Y = \{4, 5, 6, 7\}$;

- Find the formula for the relation from set X to set Y, defined by “times $\frac{1}{2}$ plus 3 is”.
- Write the relation in set builder notation.

Solution

- Let x represent a member of the domain and y represent a member of the range.

Then the relation is ‘x times $\frac{1}{2}$ plus 3 is y’, or

‘y is equal to x times $\frac{1}{2}$ plus 3’,

which gives the formula $y = \frac{1}{2}x + 3$.

- If the relation is R, then in set-builder notation,

$$R = \{(x, y): y = \frac{1}{2}x + 3, x \in X, y \in Y\}$$

You should now do the activity that follows to practise what you have just learnt. The activity is similar to the example you have just gone through.



Activity 5

The set $R = \{(-2, 0), (-1, 1), (0, 2), (1, 3)\}$ of ordered pairs represents a relation from $C = \{-2, -1, 0, 1\}$ to $D = \{0, 1, 2, 3\}$.

- Define the relation.
- If $x \in C$ and $y \in D$, write down the formula for the relation.

Write down your answer in the space provided below.



Feedback

Feedback

- (a) *When we examine the corresponding elements in set C and set D we discover that the relation that exists between them is “plus 2 is”. Therefore, to find the images, we have to add 2 to each and every element in set C.*

Remember that first numbers in the given set of ordered pairs forms the domain. These numbers are -2, -1, 0 and 1. Thus

$$-2 + 2 = 0$$

$$-1 + 2 = 1$$

$$0 + 2 = 2$$

$$1 + 2 = 3$$

- (b) *Let x represent a member of the domain and y represent a member of the range. Then the relation is ‘x plus 2 is y’, or ‘y is equal to x plus 2’, which gives the formula $y = x + 2$. Therefore, $R = \{(x, y): y = x + 2, x \in C, y \in D\}$*

So far you have learnt how to represent the relation using arrow diagrams, sets of ordered pairs and a formula. Now we will look at the last method of representing the set. The method is Cartesian Graphs

Cartesian Graphs

The Cartesian Graph derives its name from a French mathematician called Rene Descartes, who first developed this type of graph. The Cartesian Graph is usually referred to as simply a graph.

To draw the Cartesian Graph of a relation, you require two axes; the horizontal and vertical axes. The domain forms the horizontal axis and the range forms the vertical axis.

Let us look at the following example

Example 5

Represent the relation “is a prime factor of” from $A = \{2, 3, 5\}$ to $B = \{4, 6, 8, 10\}$ by a graph.

Solution

- (a) **Step 1:** We first express the relation “is a prime factor” as ordered pairs as $(2, 4), (2, 6), (2, 8), (2, 10), (3, 6)$ and $(5, 10)$.

Step 2: We then plot the numbers in Cartesian Graph.

To plot (2, 4), we first locate 2 on domain axis and then 4 on range axis. Where the two lines meet is where the point is made as you can see in the graph below. We then plot the second pair (2, 6). As we did with the first point, we will first locate 2 on the domain axis and then 6 on the range axis. Where the two lines meet is where the point is made. The same procedure is followed to plot the other ordered pairs.

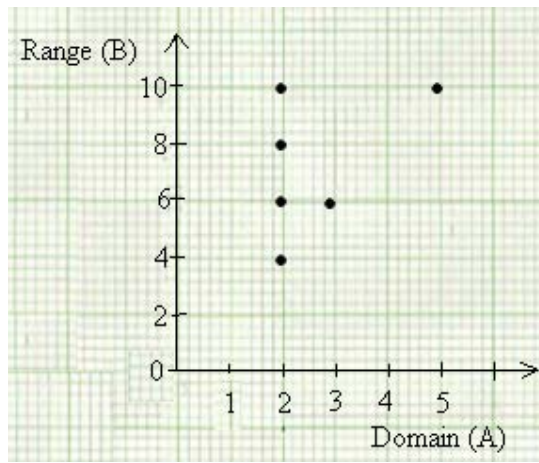


Figure 3: Cartesian Graph: “is a prime factor of”

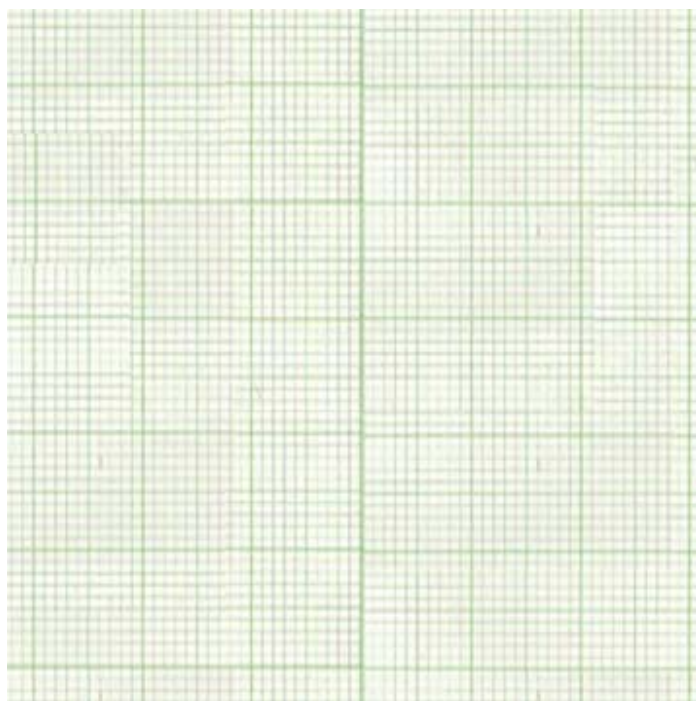
You have now learnt how to represent the relations using the Cartesian graph. You should now practice what you have just learnt by doing the following activity.



Activity 6

Consider the relation “plus 4 is” from $C = \{1, 2, 3\}$ to $D = \{5, 6, 7\}$. Represent this relation by a graph.

Use the graph below to write down your answer.





Feedback

Feedback

The ordered pairs we use to plot the graph are (1, 5), (2, 6), and (3,7).

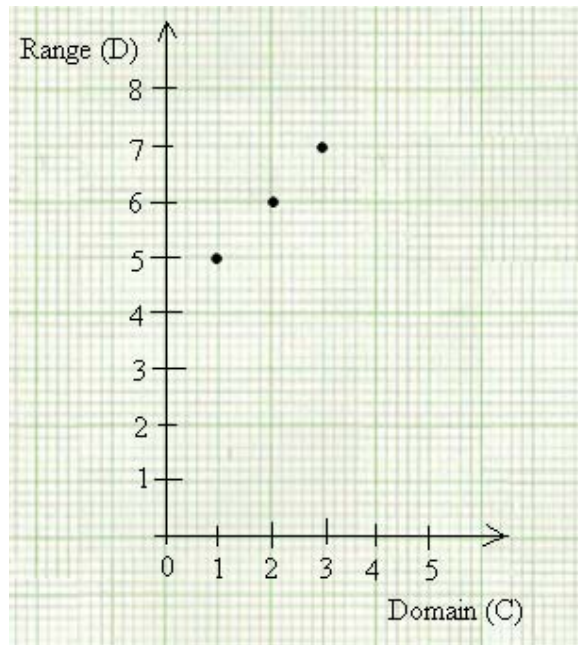


Figure 4: Cartesian Graph: “plus 4 is”

So far we have looked at four ways of representing number relations. These are arrow diagrams, sets of ordered pairs, formula and Cartesian Graphs. Now we will discuss the types of relations.

Types of Relations

Relations are grouped in four types according to the manner in which the elements of domain are paired with the elements of the range. These types are as follows:

- i. One-to-one
- ii. Many-to-one
- iii. Many-to- many and
- iv. One-to-many relations.

Let us now discuss these relations one by one;

One to one Relation

A relation is said to be a one to one if the elements are paired in such a manner that each element of the domain set is paired to only one element of the range set and each element of the range set has only one element in the domain set.

We say that the elements are in one to one correspondence and the relation is a one to one relation.

For example Sets A and B are in a one to one relation when the elements of A and B can be paired such that each element in A corresponds to exactly one element in B and each element in B has only one image in A. That is each input has only one image and each image has only one object.

Now do the following activity to practice what you have just learned.



Activity 7

The relation 'add 3' is defined on the Set $A = \{4, 5, 6, 7\}$.

- Find the image set B.
- Draw the arrow diagram in the space provided below and
- State whether the relation is 1 to 1 or not.

Write your answer in the space provided below.

Feedback



Feedback

- The first step is to add 3 to each element in set A as shown below.

$$4 + 3 = 7$$

$$5 + 3 = 8$$

$$6 + 3 = 9$$

$$7 + 3 = 10$$

Then set $B = \{7, 8, 9, 10\}$

- Using the arrow diagram we have

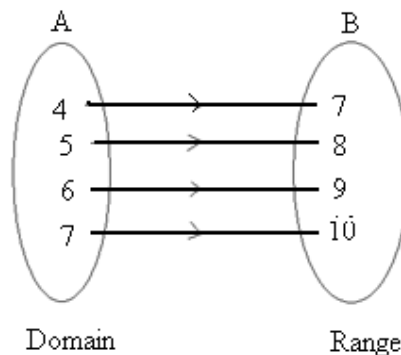


Figure 5: Arrow diagram: "add 3"

- The relation is one-to-one.

**Note**

Note that each element in set A has only one image in B and each image in B has only one object in A.

We should also note that in a one-to-one relation, the number of elements in set A must be equal to the number of elements in set B.

That is $n(A) = n(B)$.

You have learnt what one-to-one relation is. Now let us move to another type of relations. We will look at many-to-one.

Many -to-one Relation

What do you think many-to-one relation is? You saw in the previous relation that, each element in domain is paired with each element in range. Therefore, many-to-one relation is understood as a relation in which many elements in the domain are paired with one element in the range.

Before you proceed, ensure that you understand the method used in one to one relation well. For the next activity you will need to use this method as in Activity7 to work out many-to-one relation. If you are ready, do the next activity.

**Activity 8**

Illustrate the relation 'is greater than' from set $A = \{5, 6, 7\}$ to $B = \{4, 9\}$ on the arrow diagram.

Write your work in the space below and state whether the relation is many to one or not.



Feedback

Feedback

The relation “is greater than” from set A to set B is illustrated as follows:

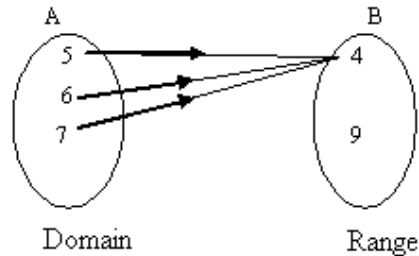


Figure 6: Arrow diagram: “is greater than”

The relation is many to one since three elements have one image, 4

So far we have looked at one-to-one relation and many-to-one relation. Now let us look at another type of relation. This time we will be looking at many-to-many relation.

Many-to-Many Relation

A relation in which at least one object has two images and at least one image has two objects is called a many to many relation. For example, the relation “is a child of” in a family of both parents with at least two children can be described as many to many; since each parent would have all children as images and all children would have both parents as their objects.

You may now work out the next activity which is similar to the previous activities. You should, therefore, find it relatively easy to complete.



Activity 9

A relation is defined as “is less than “on set $B = \{ 1,2,3,4\}$. Draw the arrow diagram to show this relation in the space below and state if the relation is many to many or not



Feedback

Feedback

Since the relation is defined on set B then both the domain and the Range are set B .

The first step is to draw two sets with the same elements. The first set, which is a domain, is named set A and the second set, which the range is named set B .

In order for us to identify the elements in set B that are less than the elements in set A , we will instead look for the elements in set B that are greater than elements in set B .

The second step is to determine the elements in set B that is greater than 1. These elements are 2, 3, and 4. Therefore, we can say that 1 is less than 2, 3 and 4.

Then we will move to the next element in set A . The next element is 2. The elements in set B that are greater than 2 are 3 and 4. We can therefore say that 2 is less than are 3 and 4.

We should move to the third element in set A which is 3. The elements in set B that are greater than 3 is 4. We can therefore say that 3 is less than 4.

You should note that 4 in set A is not paired with any element in set B because there is no element in B which is greater than 4.

Using the arrow diagram we have.

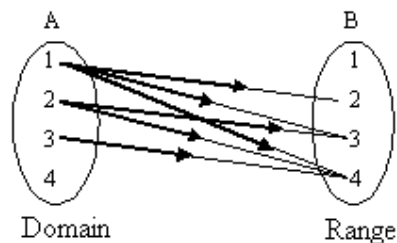


Figure 7: Arrow diagram: “is less than”

Now let us move to the last type of the relations. This is one-to-many relation.

One-to-many relation

A relation in which at least one object has more than one image is called one-to-many relation. From the discussion and activities you have done in the previous relations, you should be able to do the next activity.

You may proceed to the activity on one – to - many.



Activity 10

Mr. Banda is the father of Liseli and Pumulo, Mr. Ng’ambi is the father of Isaac, Charles and Florence. If F is the set of fathers and C is the set of children then illustrate this relation by drawing the arrow diagram from set F to set C in the space below.



Feedback

Feedback

$$F = \{Banda, Ng'ambi,\}$$

$$C = \{Liseli, Pumulo, Isaac, Florence\}$$

The relation R is “is the father of”

The arrow diagram in Figure 8 shows the relation “is the father of” from set F to set C .

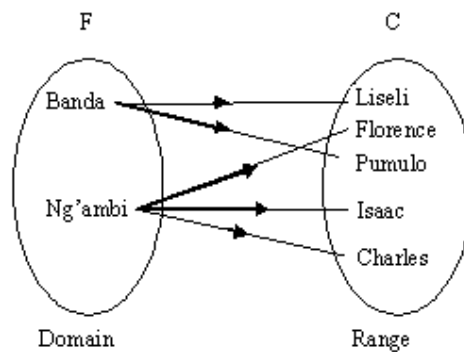


Figure 8: Arrow diagram: “is the father of”

You have just learned the four types of relations namely **one-to-one**, **many-to-one**, **many-to-many** and **one-to-many** relations. Now we can look at the topic summary and topic exercise to revise what we have covered so far.

Topic 1: Summary



Summary

In this topic you learnt what a relation means and how to represent it in different ways. We defined a relation as a rule which associates the elements of two sets. Whereas, the relation between two variables x and y is a set of ordered pairs. The x -values are the inputs, or the domain, and the y -values are the outputs, or the range.

The relations can either be number relations or non-number relations. The number relations can be represented in four ways which are arrow diagrams, sets of ordered pairs, the formula and Cartesian Graphs. There are three ways of representing non-number relations which are arrow diagrams, sets of ordered pairs and Cartesian graphs, The non-number relations cannot be represented by formulae.

In arrow diagrams, objects are connected to their images by arrows while In sets of ordered pairs, objects and their images are paired in such a way that the first element of a pair comes from the domain and the second element comes from the range. In a formula, where x represents a number of the domain and y represents a member of the range, the relation is expressed as an equation in terms of y . Finally, in Cartesian graphs, the relation is represented as a graph, the domain forms the horizontal axis and the range forms the vertical axis.

We also we looked at four types of relations. These are one-to-one relation, one-to-many relation, many-to-one relation and many-to-many relation. In one-to-one relation, each object has only one image and each image has only one object. In one-to-many relation at least one element of the domain has more than one image. In many-to-one relation, at least two elements in the domain have one image and no element in the domain has more than one image. In many-to-many relation, at least one element of domain has two images and at least one element of the range has two objects.

As part of the learning process you were also encouraged to participate in a number of activities which required you to answer all the questions in each of the activities provided in this topic. The activities were intended to help you assess how well you understood the content of the topic. If you did not do very well in the activity, this means that you need to go over the material again. Have you done this? If not, you should do this now so that you are well prepared for Topic Exercise 1.

Now you should do Topic Exercise 1 at the end of the unit to see how much you have learned. After completing the exercise, you should mark your own work by comparing your answers with those provided in the feedback soon after Topic Exercise 2. If any of your answers were incorrect, revise the relevant section/s before proceeding to Topic 2.

Topic 2: Functions

In Topic 1: Relations we looked at different ways of representing both number and non-number relations. These are arrow diagrams, sets of ordered pairs, the formula and Cartesian Graphs. We also looked at four types of relations which are one-to-one relation, one-to-many relation, many-to-one relation and many-to-many relation.

This topic is about functions. In this topic we will discuss mappings and identify the relations that are mappings and functions. Mapping will be explained in the first part of the discussion in this topic. You will also learn how to use the function notation to express the relations. We will further look at inverse functions.

As in the previous topic, you will be required to do activities within the topic which will help you to practice your skills on how to solve problems on function. You will also do Topic Exercise 2 to assess your performance. The Topic Exercise 2 is found at the end of this unit soon after Topic Exercise 1.

Upon completion of this topic you will be able to:

- *Determine* relationships that are functions.
- *Use* function notation.
- *Find* inverse of a function.



Outcomes

The learning outcomes you have just gone through will help you to know what you are expected to learn after studying through the unit.

Let us proceed with our first section in which you will learn how to determine the relations that are functions.

Determining Relationships that are Functions

In order for us to know exactly what type of relationships are functions, we need first to define what a function is.

A function can be defined as a mapping with the difference being that in a function, there is only one image for each and every member of the domain.

Then what is mapping? Of course mapping can be defined as a relation from set A to set B if each element in A is paired with exactly one element in B.



Note it!

You should note that a mapping is a relation which has the following properties:

- (a) It is either one-to-one or many- to-one relation as you can see from Figures 1 (a) and (b).

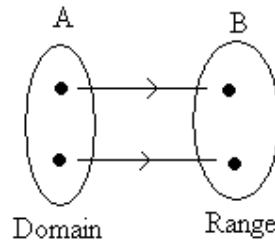


Figure 1(a): One-to-one relation

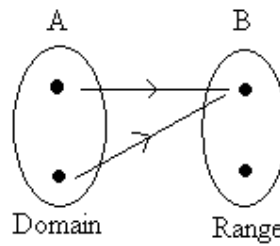


Figure 1(b): Many-to-one relation

- (b) Every element of the domain has an image.

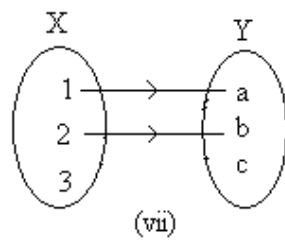
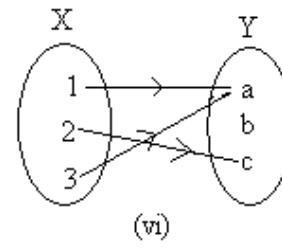
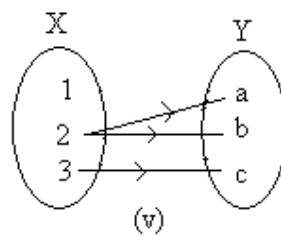
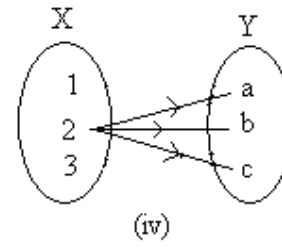
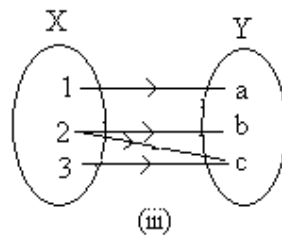
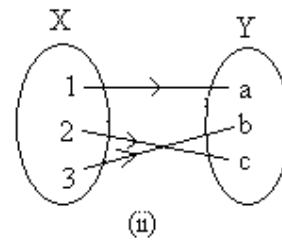
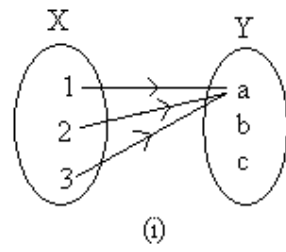
Now you should do the following activity to practice what you have just learnt.



Activity 1

The diagrams (i) to (vii) show relations from set X to set Y , where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

- (a) Which of these relations are functions?
 (b) Give reasons for those that are not functions.





Feedback

Feedback

- (a) *The relations that are functions are (i), (ii) and (vi). They are functions because every element of the domain has an image*
- (b) (iii) *Not a function because: 2 has two images*
- (iv) *Not a function because: 2 has three images and 1 and 3 have no images*
- (v) *Not a function because: 2 has two images and 1 has no image*
- (vii) *Not a function because: three has no image*

So far we discussed what a function is and the relationships that are functions. Now we can proceed to the next section where we will discuss how to use function notation.

Using Function Notation

To discuss how to use function notation we need to consider other definition of function. Function can also be defined as a correspondence or rule that assigns to every element in a set D exactly one element in a set R . The set D is called the **domain** of the function, and the set R is called the **range**.

The diagram below shows a function f mapping, or pairing, a domain element x to a range element $f(x)$.

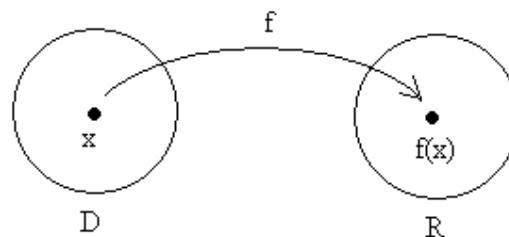


Figure 2: Function

We read $f(x)$ as “the value of f at x ” or “ f of x .”

Although f names the function and $f(x)$ gives its value at x , we sometimes refer to the function $f(x)$, thereby indicating both the function f and the variable x of its domain.

We can treat a function f as a set of ordered pairs (x, y) such that x is an element of the domain of f and y is the corresponding element of the range. This is written formally as $\{(x, y): y = f(x)\}$ or more simply as

$$y = f(x)$$



Note it!

You should note that although the letters f , x and y are commonly used in general discussions of functions, other letters can be used. For example, $v = g(u)$ is a function g that assigns a domain element u to a range element v .

A function is frequently given in terms of a **rule** and **domain**. If the domain of a function is not specified, then it is understood to consist of those real numbers for which the function produces real values.

As we have already stated, a function is usually denoted by f . If a function f maps an element x of a set A onto an element y of a set B , then this is written as

$f: x \rightarrow y$; read as “ f maps x onto y ”

The function can be expressed as the formula as

$f(x) = y$; read as “ f of x is equal to y ”

The element x is the object and y is the image. In the notation

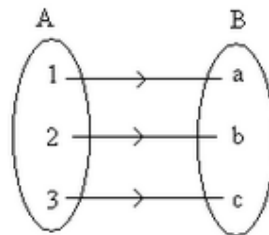
$f: x \rightarrow y$ or $f(x) = y$, y is usually expressed in terms of x



Note it!

A function being a relation, it can be expressed in four different ways. These are as follows:

(i) Arrow diagram:



(ii) Functional notation: $f: x \rightarrow y$

(iii) Formula: $f(x) = y$

(iv) Set builder notation: $f = \{(x, y), y = f(x)\}$

Now we hope that at this point you understand what a function is and are able to express it in different ways. The activity that follows will help you to practice what you have just learnt.



Activity 2

If $P = \{a, b\}$ and $Q = \{x, y\}$, use arrow diagrams to construct four functions from set P to Q .

Use the space below to write down your answer.



Feedback

Feedback

The arrow diagram:

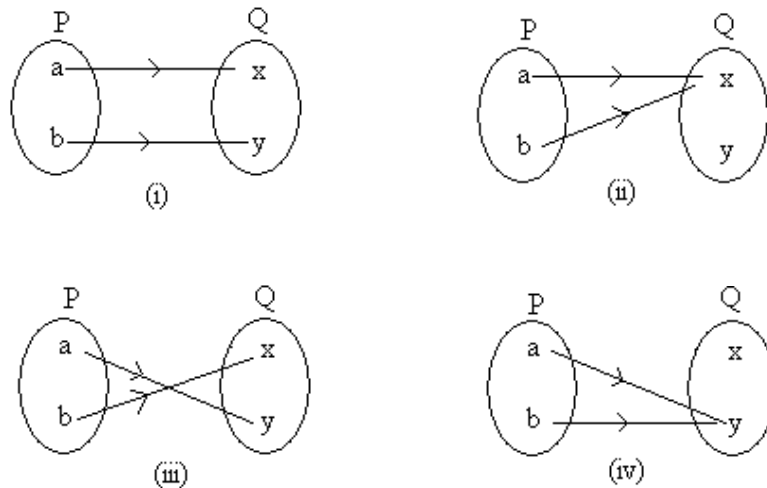


Figure 3: function as arrow diagrams

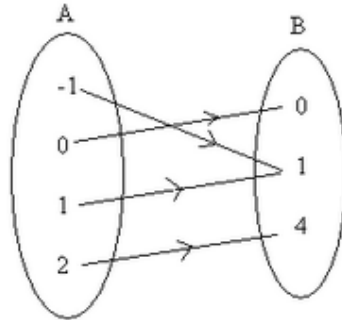
We hope that you found the activity quite interesting.

We have already pointed that the function can be represented in four different ways namely; arrow diagram, formula, functional notation and set builder notation. We have just discussed and practised using the

arrow diagram. Let us look at the following example which is about expressing the function in different ways.

Example 1

Study the arrow diagram that follows.



The arrow diagram shows a function from set A to set B.

- (a) Find the formula which defines the function.
- (b) Express the function using the following notations:
 - (i) $f: x \rightarrow y$
 - (ii) $f(x) = y$
 - (iii) Set builder notation

Solution

- (a) The elements in B are obtained by squaring the elements in A. So if $x \in A$ and $y \in B$, the relation is “x squared is y” or “y is equal to x^2 ” which gives the formula $y = x^2$
- (b) The function can be expressed as
 - (i) $f: x \rightarrow y$ gives $f: x \rightarrow x^2$
 - (ii) $f(x) = y$ gives $f(x) = x^2$
 - (iii) Set builder notation gives

$$f = \{(x, y), y = x^2, x \in A, y \in B\}$$

Now you have learnt how to express the function in different ways. We will now discuss how to find the range.

As we earlier pointed out that a function is frequently given in terms of a **rule** and **domain**, we will use the rule and the domain to find the range. Now let us do following example together. In this example we will find the range based on the given domain and the rule.

Example 2

Find the image of the function f defined as “multiply by 2 then add 1” on the set $A = \{3, 4, 5\}$.

Solution

To find the image of the function we follow the following steps:

Step 1: We represent the relation as the formula

$$f(x) = 2x + 1$$

Step 2: Our domain for the function is set $A = \{3, 4, 5\}$. We substitute the x -values in domain as follows

$$f(3) = 2(3) + 1 = 6 + 1 = 7$$

$$f(4) = 2(4) + 1 = 8 + 1 = 9$$

$$f(5) = 2(5) + 1 = 10 + 1 = 11$$

Therefore, the set of the images is $\{7, 9, 11\}$.

Now you should practice how to find the range by doing the following activity. We hope that you will not have any problem doing the activity since the activity is similar to the example we have just gone through. The only different you will find is that in our example we had 3 objects in the domain but in the following activity you have only one object.

**Activity 3**

Find the image of 6 given that $f(x) = 3x + 2$. Write your answer in the space provided below.



Feedback

Feedback

We will substitute 6 for x in the formula as follows:

$$f(x) = 3x + 2$$

$$f(6) = 3 \times 6 + 2$$

$$= 18 + 2$$

$$= 20$$

We substitute x for 6 in the formula

We simplify

Therefore, the image of 6 is 20.

You have just learnt how to find the range based on the given rule and domain. Now we will discuss how to find the domain or the objects based on given rule and the range. Let us do the following example.

Example 3

For the function $f(x) = 3x + 4$, the image is 13 find the object.

Solution

For the given function $f(x) = 3x + 4$, the object element is x and the image element is $(3x + 4)$. Since the image of the function is 13 but also $3x + 4$ then we have:

$$3x + 4 = 13$$

$$3x = 13 - 4$$

$$3x = 9$$

$$x = \frac{9}{3}$$

$$x = 3$$

Add - 4 on both sides of the equals

Divide both sides by 3

Therefore, the object for the image is 3.

In the example we have just gone through, the numbers 3 and 4 in the image element $(3x + 4)$ are referred to as constant since they do not change whenever we substitute the values of x in the formula.

Now let us discuss how to calculate the constant given at least two objects and their images and the rule.

Example 4

If $f(x) = ax + b$, where a and b are constant and $f(1) = 6$, $f(-2) = 3$

(a) find a and b

(b) write the function

Solution

(a) To find the value of a and b, we have to construct two equations and solve them simultaneously.

Step 1: We construct the first equation using $f(1) = 6$ and the formula $f(x) = ax + b$, where $x = 1$ and $f(x) = 6$.

$$f(x) = ax + b$$

$$6 = a(1) + b$$

$$6 = a + b \dots \dots \dots \text{equation (1)}$$

We label this as equation (1)

Step 2: We construct the second equation using $f(-2) = 3$ and the formula $f(x) = ax + b$, where $x = -2$ and $f(x) = 3$.

$$f(x) = ax + b$$

$$3 = a(-2) + b$$

$$3 = -2a + b \dots \dots \dots \text{equation (2)}$$

We label this as equation (2)

Step 3: We solve the simultaneous equations (1) and (2), by subtracting (2) from (1).

$$(a + b) = 6 \dots \dots \dots (1)$$

$$- (-2a + b = 3) \dots \dots \dots (2)$$

$$(a + b) - (-2a + b) = 6 - 3$$

$$a + b + 2a - b = 3$$

$$3a = 3$$

$$a = 1$$

We open the brackets

We simplify the like terms

We know that $a + b = 6$ then we substitute 1 for a.

Therefore, $1 + b = 6$

$$b = 6 - 1$$

$$b = 5$$

Therefore, $a = 1$ and $b = 5$.

(b) Replacing **a** by 1 and **b** by 5 in the function, $f(x) = ax + b$, we get:

$$f(x) = x + 5$$

From the example we have just gone through, you have learnt how to calculate the values of the constants given the two objects and their images and the rule. You should do the following activity to practice what you have just learnt.



Activity 4

Given that $f(x) = px + q$ where p and q are constants and that $f(1) = -2$ and $f(3) = 6$, find

- (a) The value of p and q
- (b) The function in terms of x .

Write your answer in the space below.



Feedback

Feedback

We hope you remembered to apply the steps taught in the example above. These are:

(a) Step 1: We construct the first equation using $f(1) = -2$ and $f(x) = px + q$ where $x = 1$ and $f(x) = -2$.

$$f(x) = px + q$$

$$-2 = 1 \times p + q$$

$$-2 = p + q \dots\dots\dots \text{equation (1)}$$

$$p + q = -2$$

We can rearrange the equation

Step 2: We construct the second equation using $f(3) = 6$ and $f(x) = px + q$ where $x = 3$ and $f(x) = 6$.

$$f(x) = px + q$$

$$6 = 3 \times p + q$$

$$6 = 3p + q \dots\dots\dots \text{equation (2)}$$

$$3p + q = 6$$

We can rearrange this equation as well

Step 3: We solve the two equations simultaneously by subtracting equation (2) from equation (1).

$$\begin{aligned} p + q &= -2 \dots\dots\dots (1) \\ -(3p + q &= 6) \dots\dots\dots(2) \end{aligned}$$

$$p + q - (3p + q) = -2 - 6$$

$$p + q - 3p - q = -8$$

$$-2p = -8$$

$$P = \frac{-8}{-2}$$

$$P = 4$$

Divide both sides by -2

Then we substitute 4 for p in equation (1)

$$p + q = -2$$

$$4 + p = -2$$

$$q = -2 - 4$$

$$q = -6$$

Therefore, $p = 4$ and $q = -6$.

(b) Replacing p by 4 and q by -6 in the function, $f(x) = px + q$, we get:

$$f(x) = 4x - 6$$

So far in this topic we have looked at how to determine the relationships that are functions and how to use function notation. We can now proceed to the last section of our study.

In the next section we will discuss how to find the inverse a function.

Inverse Functions

The knowledge of inverse functions is important in real-life situations especially in the conversion formulas. One of the examples we can give is temperature conversion. The formula to convert temperature in degree Celsius to temperature in degrees in Fahrenheit is:

$$F = \frac{9}{5}C + 32$$

For this formula, C is the input and F is the output. We rewrite the formula so that F in the input and C is the output. Of course the formula can be written as follows

$$C = \frac{5}{9}(F - 32)$$

The first formula gives a Fahrenheit temperature F as a function of a Celsius temperature of F .

Notice that in the first formula, $F = 32$ when $C = 0$, and in the second formula, $C = 0$ when $F = 32$. Because each formula undoes what the other one does the formula are examples of inverse.

To discuss the inverse functions let us begin by describing how to find the inverse relation of the relation given by a set of ordered pairs and an equation in x and y .

Inverse of Relation

The inverse of a relation is the set of ordered pairs obtained by switching the coordinates of each ordered pair in the relation.

Here is an example.

$\{(0, -3), (1, -1), (2, 1), (3, 3), (4, 5)\}$ Original relation

$\{(-3, 0), (-1, 1), (1, 2), (3, 3), (5, 4)\}$ Inverse relation



Note it!

You should note that to find the inverse of the relation, we switch the (x, y) . For example the first ordered pair $(0, -3)$ where $x = 0$ and $y = -3$ becomes $(-3, 0)$ where $x = -3$ and $y = 0$.

You should also note that the graph of the inverse is the reflection of the graph of the original relation. The mirror of the reflection is the line $y = x$.

Let us construct the graph of original relation. We will plot the point of the ordered pairs of the original relation. The first point is $(0, -3)$, the second point is $(1, -1)$ and so forth. Then we join the points we have plotted to get a straight line.

We will as well plot a mirror line $y = x$. We need any two x -values and y -values that satisfy the equation $y = x$. The two points we can use are $(3, 3)$ and $(1, -1)$.

The figure below shows the graphs for $y = 2x - 3$ and $y = x$.

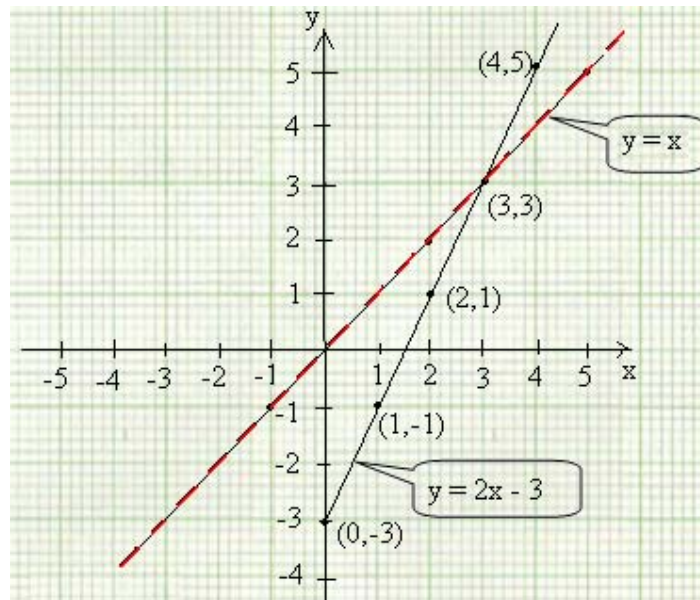


Figure 3(a): Graph of Original Relation

Let us construct the graph of inverse relation. After drawing the axes, we then plot the points represented by the sets of ordered pairs $\{(-3, 0), (-1, 1), (1, 2), (3, 3), (5, 4)\}$. You should remember that the first numbers in the ordered pairs have to read on x-axis and the second numbers are to be read on the y-axis. We start with the first point $(-3, 0)$. Note that this point is on x-axis where $x = -3$. We then move to plot other points as we did in the previous graph.

We will again draw the graph $y = x$ as mirror line of our reflection. The graphs of inverse relation are shown in the figure below.

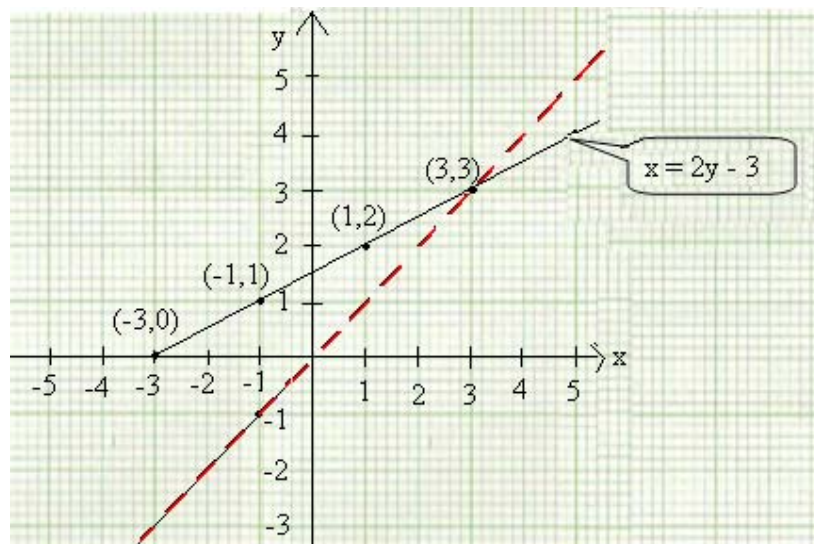


Figure 3(b): Graphs of Inverse Relation

Since the graphs we have just constructed above are the inverse of the other, we can therefore draw them on the same graphs and see below how they will appear.

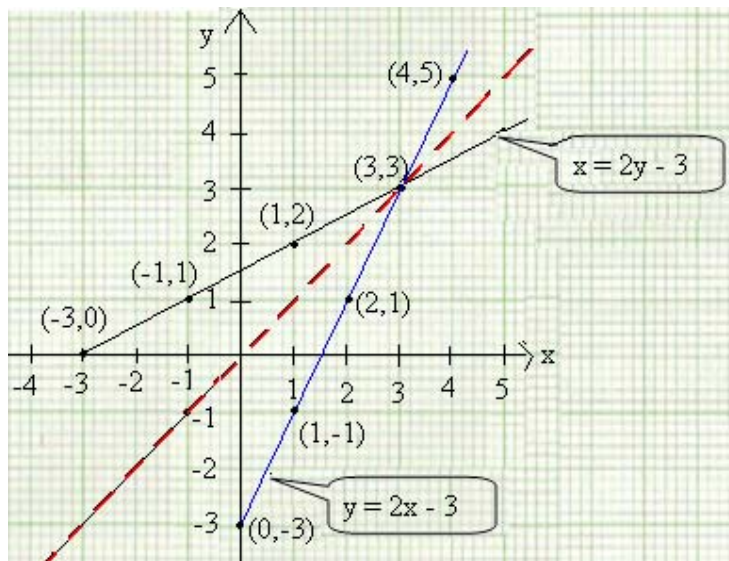


Figure 3(c): The graphs of $y = 2x - 3$ and $x = 2y - 3$ are reflections of each other in the line $y = x$

Let us do the following example in which we will discuss how to find the inverse of the relation.

Example 5

Find the equation for the inverse of the relation $y = 2x - 3$.

Solution

$y = 2x - 3$ original equation

As we earlier noted, we will switch the roles of x and y to find an equation for the inverse relation.

$x = 2y - 3$

Note the new position of x and y

Be sure you see that these two equations are not equivalent. When finding an equation of an inverse relation, you are forming a different

equation whose solutions reflect the switching of x and y . You can confirm this by sketching their graphs.

The inverse of a function is found in the same way you found the inverse of a relation (because a function is simply a special type of relation). It is important to realize, however, that the inverse of a function may not itself be a function.

The figure below illustrates the inverse of the function.

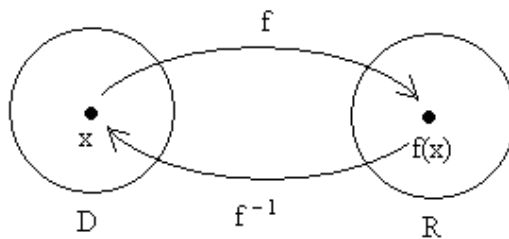


Figure 4: inverse function

Let us do the next example in which will find the inverse of the function.

Example 6

Find the inverse of the function $f(x) = 3x - 1$.

Solution

To begin, replace $f(x)$ by y , then switch x and y in the equation

$f(x) = 3x - 1$	Original function
$y = 3x - 1$	Replace $f(x)$ by y
$x = 3y - 1$	Switch x and y to obtain inverse
$3y = x + 1$	Expressing the equation in terms of y
$y = \frac{x+1}{3}$	
$y = \frac{1}{3}x + \frac{1}{3}$	

We denote the inverse of a function f by f^{-1} , read “ f inverse.” The symbol $f^{-1}(x)$, read “ f inverse of x ,” is the value f^{-1} at x . Note that $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$.

$$f^{-1}(x) = \frac{1}{3}x + \frac{1}{3} \quad \text{inverse function}$$



Note it!

Note that to find the inverse of a function we:

- Interchange x and y in the equation of the function.
- Solve the resulting equation for y .

Now you have learnt how to find the inverse function. You should do the following activity which will help you to understand how to find the inverse function. .

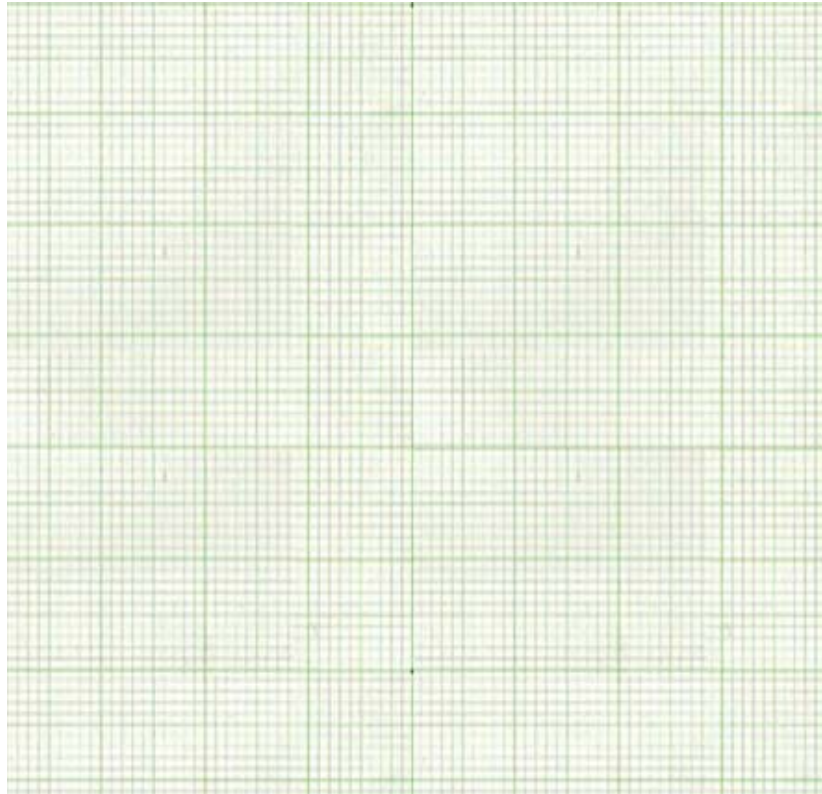


Activity 5

Sketch the following function and its inverse in the same coordinate plane.

$$f(x) = 3x + 4$$

Use the space and the graph provided below to write down your answer.



Feedback

Feedback

$$f(x) = 3x + 4 \quad (\text{Original function})$$

$$y = 3x + 4 \quad (\text{Replace } f(x) \text{ by } y)$$

$$x = 3y + 4 \quad (\text{switch } x \text{ and } y \text{ to obtain inverse})$$

$$3y = x - 4$$

$$y = \frac{x - 4}{3}$$

$$y = \frac{1}{3}x - \frac{4}{3} \quad \text{where } y = f^{-1}(x) \text{ then}$$

$$f^{-1}(x) = \frac{1}{3}x - 4$$

Let us now draw the graph of the function $f(x) = 3x + 4$ and inverse function $f^{-1}(x) = \frac{1}{3}x - \frac{4}{3}$

Graph of function $f(x) = 3x + 4$

Step 1: we first replace $f(x)$ by y and then find the y -intercept, let $x = 0$.

$$y = 3x + 4$$

$$y = 3(0) + 4$$

$$y = 4$$

The line passes through $(0, 4)$, so the y-intercept is 4

To find the x-intercept, let $y = 0$

$$y = 3x + 4$$

$$0 = 3x + 4$$

$$3x = -4$$

$$X = -\frac{4}{3}$$

$$X = -1\frac{1}{3}$$

The line passes through $(-1\frac{1}{3}, 0)$, so the x-intercept is $-1\frac{1}{3}$.

Step 2: We plot $(0, 4)$ and $(-1\frac{1}{3}, 0)$. Draw a straight line through them.

Graph of inverse function $f^{-1}(x) = \frac{1}{3}x - \frac{4}{3}$

Step 1: replace $f^{-1}(x)$ by y

$$y = \frac{1}{3}x - \frac{4}{3} \text{ To find the y-intercept, let } x = 0.$$

$$y = \frac{1}{3}(0) - \frac{4}{3}$$

$$y = -\frac{4}{3}$$

The line passes through $(0, -\frac{4}{3})$, so the y-intercept is $-\frac{4}{3}$ or -1.3

To find the x-intercept, let $y = 0$

$$y = \frac{1}{3}x - \frac{4}{3}$$

$$0 = \frac{1}{3}x - \frac{4}{3}$$

$$\frac{1}{3}x = \frac{4}{3}$$

$$X = \frac{4}{3} \times 3$$

$$X = 4$$

The line passes through $(4, 0)$, so the x-intercept is 4.

Step 2: Plot $(0, -1.3)$ and $(4, 0)$. Draw a straight line through them.

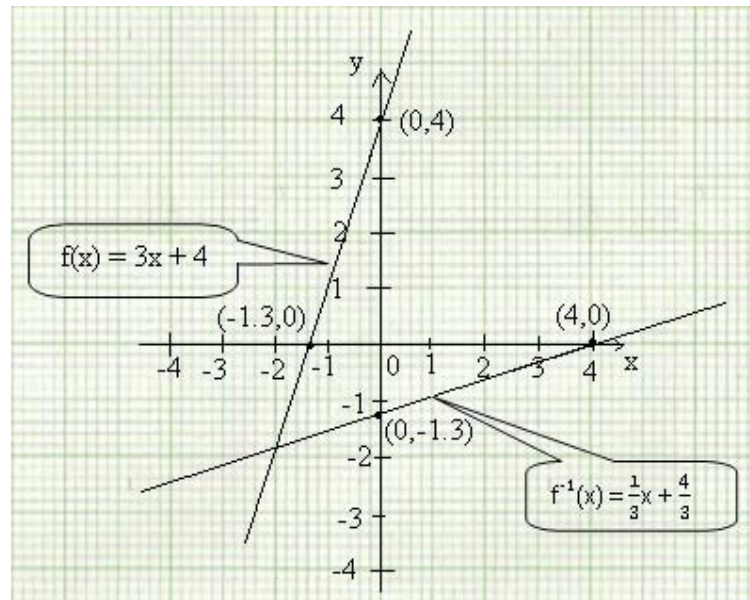


Figure 5: Graphs of $f(x) = 3x + 4$ and $f^{-1}(x) = \frac{1}{3}x - \frac{4}{3}$

So far we have seen that when certain changes are made in the equation of a function, there is a corresponding change in its graph. We shall now investigate the effect of interchanging x and y in the equation of a function.

Consider the linear function $y = 3x + 2$. If x and y are interchanged, we get $x = 3y + 2$.

Solve for y : $x = 3y + 2$.

$$3y = x - 2$$

$$y = \frac{x-2}{3}$$

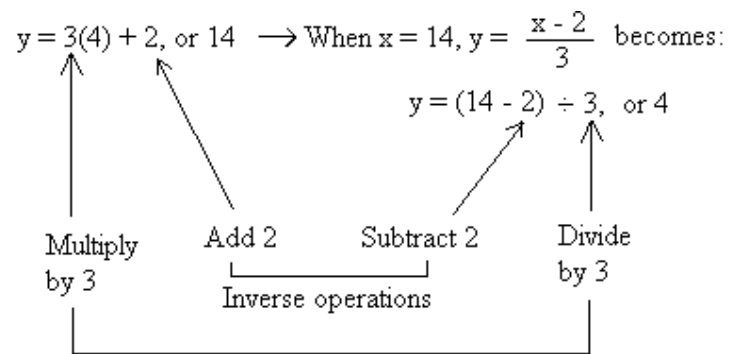
$$y = \frac{1}{3}x - \frac{2}{3}$$

This is also linear function.

Let us use **Algebraic Comparison to** show that $y = \frac{1}{3}x - \frac{2}{3}$ is an inverse function of $y = 3y + 2$.

Let x be any number, for example, 4.

When $x = 4$, $y = 3x + 2$ becomes:



The function $y = \frac{x-2}{3}$ is called the inverse of the function $y = 3x + 2$.



Note it!

As we can see from our illustration above,

- multiplication and division are inverse operation
- Addition and subtraction are inverse operation

You have come to the end of topic 2. We hope that you enjoyed working through this topic and you felt motivated to take part in doing the activities in order to strengthen your skills in applying some of the new knowledge which you acquired from our discussions. Now we would like to invite you to read through a topic summary that follows.

Topic 2: Summary



Summary

In Topic 2: Functions, you learned how to identify the relations that are mappings and functions. You also learnt how to represent functions as arrow diagram, functional notation, and formula and Cartesian graph. You further learnt how to find the inverse of the function.

As part of the learning process you were also encouraged to participate in a number of activities which required you to answer all the questions in each of the activities provided in this topic. The activities were intended to help you assess how well you understood the content of the topic. If you did not do very well in the activity, this means that you need to go over the material again. Have you done this? If not, you should do this now so that you are well prepared for Topic Exercise 1.

Now you should do Topic Exercise 2 at the end of the unit to see how much you have learned. After completing the exercise, you should mark your own work by comparing your answers with those provided in the feedback soon after the answers to Topic Exercise 1. If any of your answers were incorrect, revise the relevant section/s before proceeding to Unit 3.

You will remember that this Unit had only two topics. This therefore means that we have now come to the end of our discussions on Unit 2. To conclude this we would like to invite you to read the Unit Summary.

Unit Summary



Summary

Congratulations for having completed this unit. In Topic 1: Relations, you learnt how to represent number and non-number relations using arrow diagrams, sets of ordered pairs, formula and Cartesian graph. You also learnt the types of relations which include one-to-one relation, one-to-many, many-to-one, and many-to-many. You also learned how to find images of the objects for a given relation.

In Topic 2: Functions, you learned about mappings and functions. You learnt how to represent the function using arrow diagrams, the sets of ordered pairs, functional notation and Cartesian graph.

In this unit, like in the previous units, you were encouraged to answer all the questions in the activities. The activities were intended to help you assess how well you understood the content of the topics. You should now be well prepared to do the last topic exercise which you will find at the end of the unit. If you do not do very well in the topic exercise, this means that you need to go over the material again. You will be asked to complete and submit a tutor-marked assignment at the end of Unit 3.

The next unit, Unit 3, is on Graphs Of Polynomials. You will learn how to draw these graphs. You will also learn the application of graphs.

We hope you will enjoy studying Unit 3 as well.

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Topic Exercise 1



Exercise

Answer the following questions on a separate answer sheet.

- A relation is defined with the domain set $D = \{\text{Nairobi, Kampala, Harare, Cairo}\}$ find the range set R if the relation is “capital city of”
- A relation has domain set $\{4, 8, 12, 16\}$, find the range set of the relation “a quarter of”
- Mr. Kabaso has a set of A of his sons and a set B of his daughters. State a possible relation from
 - Set B to set A
 - Set A to set B
- A relation has range as $\{8, 10, 12, 14\}$. Find the domain set if the relation is “plus 2 is”
- What relation is illustrated in the arrow diagram below?

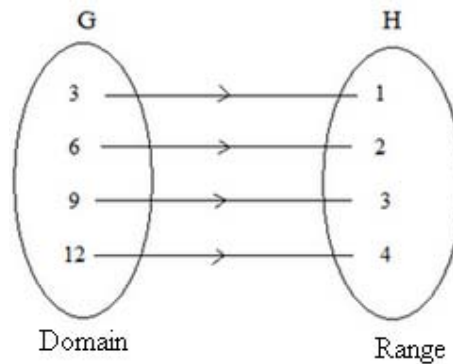
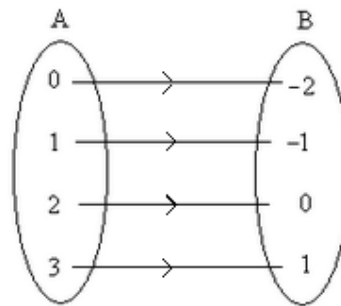


Figure 6: Relation from G to H

- Use the arrow diagram to represent the relation “is a prime factor of” from $A = \{2, 3, 5\}$ to $B = \{4, 6, 8, 10\}$.
[You may recall factors and prime factors in your junior course. For example 7 is a prime factor of 14].
- The arrow diagram represents a relation from set A to set B .



Define the relation.

- (a) If $x \in A$ and $y \in B$, write down the formula for the relation.
8. A relation is given by the set R of ordered pairs. $R = \{(-2, 3), (-1, 4), (1, 6), (2, 7)\}$.
- (a) Write down the domain and range of R.
- (b) Define the relation represented by R.
- (c) Express the relation R in set builder notation.
- (d) What type of relation is R?
9. The relation “is a factor of” is defined on the set $Q = \{1, 2, 3, 4, 5\}$.
- (a) Draw an arrow diagram to show this relation.
- (b) Express the relation as a set of ordered pairs.
- (c) Write down the domain and range of the relation.
- (d) What type of relation is it?
10. $X = \{3, 4, 5, 6\}$. A relation on the set X is defined by “is greater than”.
- (a) Express this relation as a set of ordered pairs.
- (b) Write down the domain and range.
- (c) Draw the graph of the relation.
- (d) What type of the relation is it?

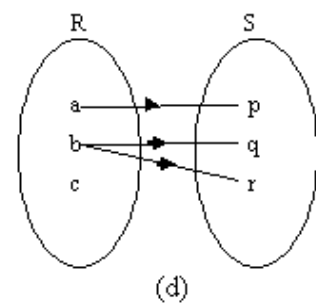
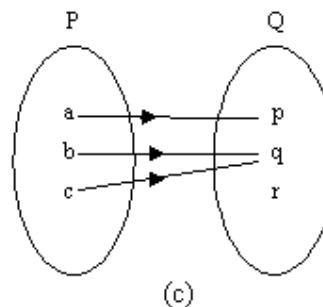
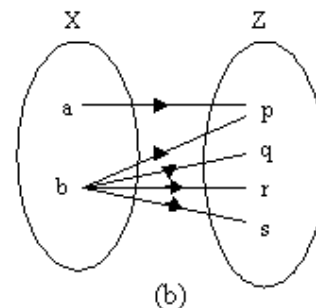
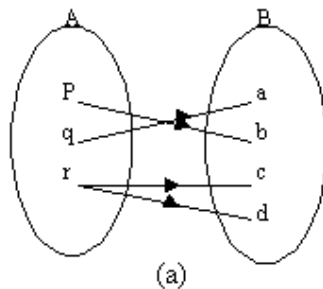
You have come to the end of Topic Exercise 1. You might have found the exercise challenging and interesting. You should now mark your work by comparing your answers with those provided at the end of Topic Exercise 2 of this unit.

Topic Exercise 2



Exercise

1. The diagram marked (a) to (d) represent some relations. State which are mappings and which are not and give reasons why in the space below.



- The relation "is a factor of" is defined on set $A = \{2, 3, 4, 5, 6\}$. Is the relation on A a mapping? Give a reason why.
- The relation R : "multiply by 2 and then subtract 1" is defined on set $A = \{3, 4, 5, 6\}$. Find the range, B and state whether the relation R' is a mapping.
- Given that $f(x) = 3x - 2$, where $x \in \mathbb{R}$ (x is a real number).
Find (a) $f(1)$
(b) $f(-1)$
- Find the input x if $f(x) = 3x - 1$ has 11 as the image element.
- Given that $f(x) = px + q$ where p and q are constants, find p and q if $f(3) = 11$ and $f(1) = 7$. Hence write the function in terms of x .
- Given that $g : x \rightarrow 2x^2 - 3x + 5$ has domain $\{-1, 0, 1, 2, 3\}$, find the range.
- The sum of the interior angles of a regular polygon of n sides is given by the formula $S(n) = (2n - 4) \times 90^\circ$

- a) State the(i) input element and (ii) the outputs of the inputs
 - b) Find the sum of the angles of a regular polygon with 21 sides.
 - c) How many sides has a regular polygon whose sum of angles is 2340°
9. Find the inverse function of each of the following
- a) $f(x) = 5x - 8$
 - b) $f(x) = \frac{3x-2}{4}$
10. Sketch the function and its inverse in the same coordinate plane.
- $$f(x) = x + 3$$

You have come to the end of Topic Exercise 2. You might have enjoyed the exercise especially when you had spent more time studying your course. You should now mark your work by comparing your answers with those provided at the end of this unit.

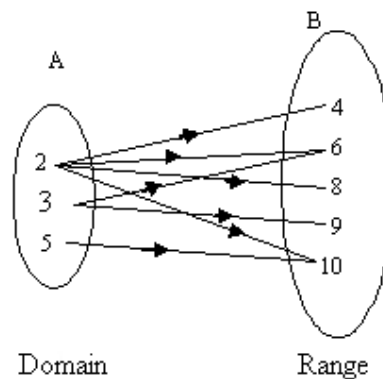
Answers for Unit 2 Topic Exercises

Answers to Topic Exercise 1

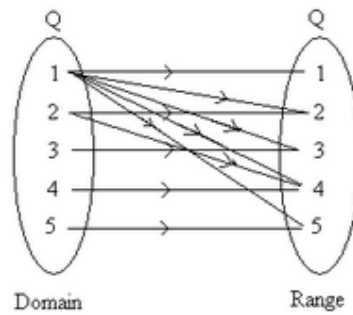


answers

1. $R = \{ \text{Kenya, Uganda, Zimbabwe, Egypt} \}$
2. $R = \{ 16, 32, 48, 64 \}$ from
3. (a) “is a sister of “
(b) “is a brother of”
4. $\{ 6, 8, 10, 12 \}$
5. “multiply by $\frac{1}{3}$ “ or “divide by 3”
6. “Is a prime factor of”



7. (a) The relation is “minus 2 is”
(b) If x represent a member in the domain and y represent a member in the range.
Then ‘ x minus 2 is y ’ or ‘ y equal to x minus 2’,
Therefore, the formula is $y = x - 2$
8.
 - (a) Domain is $\{-2, -1, 1, 2\}$; Range is $\{3, 4, 6, 7\}$
 - (b) The relation represented by R is “plus 5 is”
 - (c) If x represent a member of the domain and y represent a member of the range, then x plus 5 is y or y equal to x plus 5 which gives $y = x + 5$. If the relation is R , then in set builder notation $R = \{(x, y): y = x + 5, x \in \text{Domain}, y \in \text{Range}\}$
 - (d) The relation is one-to-one
9.
 - (a)



(b) The set of ordered pairs is

$\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 4), (3, 3), (4, 4), (5, 5)\}$

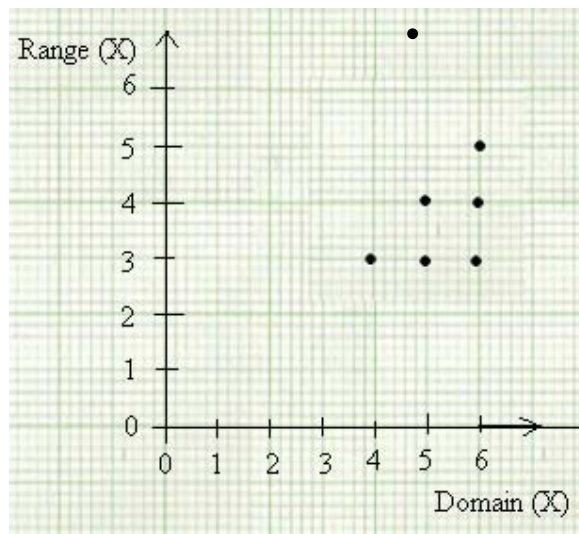
(c) Domain is the set $Q = \{1, 2, 3, 4, 5\}$; Range is the set $Q = \{1, 2, 3, 4, 5\}$.

(d) The relation is many-to-many

10. (a) $\{(4,3), (5,3), (5,4), (6,3), (6,4), (6,5)\}$

(b) Domain is the $X = \{3, 4, 5, 6\}$; Range is set $X = \{3, 4, 5, 6\}$

(c)



(d) The relation is many-to-many

Feedback

We hope that after comparing your answers with the model answers provided above, you might have got all the answers correct. Congratulations! If not then I suggest you try the questions you got wrong again. You may be required to read through the section again in order to gain more understanding.

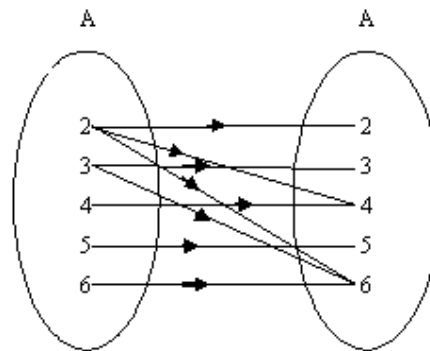
Answers to Topic Exercise 2



Answers to Assignment

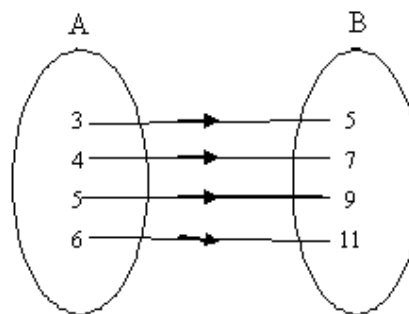
1. (a) Not a mapping since element r has two images.
 (b) Not a mapping since element b has more than one image.
 (c) Is a mapping since all elements of the domain have one image each in the range.
 (d) Is not a mapping since c has no image and b has more than one images.
2. The relation is not a mapping since elements 2 and 3 have more than one image?

You may refer to the arrow diagram



“is a factor of”

3. $A = \{ 3, 4, 5, 6 \}$ $B = \{ 5, 7, 9, 11 \}$



The relation is one to one and is a mapping.

4. (a) $F(x) = 3x - 2$
 $F(1) = 3(1) - 2$
 $= 3 - 2$
 $= 1$
- (b) $f(x) = 3x - 2$

$$\begin{aligned}
 F(-1) &= 3(-1) - 2 \\
 &= -3 - 2 \\
 &= -5
 \end{aligned}$$

5. $f(x) = 3x - 1$

$$3x - 1 = 11$$

$$3x = 11 + 1$$

$$\frac{3x}{3} = \frac{12}{3}$$

$$x = 4$$

6. Given $f(x) = px + q$ and $f(3) = 11$ then $x = 3$ and $f(x) = 11$

$$f(x) = px + q$$

$$11 = 3p + q$$

$$3p + q = 11 \text{ equation (i)}$$

For the same function $f(x) = px + q$ given $f(1) = 7$ then $x = 1$ and $f(x) = 7$

$$f(x) = px + q$$

$$7 = p + q$$

$$p + q = 7 \text{ equation (ii)}$$

Then we subtracting equation (ii) from equation (i)

$$3p + q = 11 \text{ (1)}$$

$$-(p + q = 7) \text{(ii)}$$

$$3p + q - (p + q) = 11 - 7$$

$$3p + q - p - q = 4$$

$$2p = 4$$

$$p = \frac{4}{2}$$

$$p = 2$$

Substituting the value of p in equation (ii) gives

$$2 + q = 7$$

$$q = 7 - 2$$

$$q = 5$$

The function, therefore with values of p and q is;

$$f(x) = px + q$$

$$f(x) = 2x + 5$$

$$7. \quad g : x \rightarrow 2x^2 - 3x + 5$$

$$\text{When } x = -1, g : -1 \rightarrow 2(-1)^2 - 3(-1) + 5 = 10$$

$$\text{When } x = 0, g : 0 \rightarrow 2(0)^2 - 3(0) + 5 = 5$$

$$\text{When } x = 1, g : 1 \rightarrow 2(1)^2 - 3(1) + 5 = 2 - 3 + 5 = 4$$

$$\text{When } x = 2, g : 2 \rightarrow 2(2)^2 - 3(2) + 5 = 8 - 6 + 5 = 7$$

$$\text{When } x = 3, g : 3 \rightarrow 2(3)^2 - 3(3) + 5 = 18 - 9 + 5 = 14$$

Therefore, the range is $\{4, 5, 7, 10, 14\}$ ◀

$$8. \quad S(n) = (2n - 4)90^\circ$$

(a) (i) input : number of sides

(ii) Output: sum of interior angles

$$(b) \quad n = 21, s(21) = (2 \times 21 - 4)90^\circ$$

$$= (42 - 4)90^\circ$$

$$= 38 \times 90^\circ$$

$$= 3420^\circ$$

$$(c) \quad 90^\circ (2n - 4) = 2340^\circ$$

$$2n - 4 = \frac{2340}{90}$$

$$2n - 4 = 26$$

$$2n = 26 + 4$$

$$n = 30$$

$$n = \frac{30}{2}$$

Therefore, the polygon has **15 sides**

$$9. \quad f(x) = 5x - 8$$

$$y = 5x - 8$$

$$x = 5y - 8$$

$$5y = x + 8$$

$$y = \frac{x+8}{5}$$

$$f^{-1}(x) = \frac{x+8}{5}$$

Interchange x and y

$$(b) \quad f(x) = \frac{3x-2}{4}$$

$$y = \frac{3x-2}{4}$$

$$x = \frac{3y - 2}{4}$$

$$4x = 3y - 2$$

$$3y = 4x + 2$$

$$y = \frac{4x + 2}{3}$$

$$f^{-1}(x) = \frac{4x + 2}{3}$$

10. Let us find the inverse function

$$f(x) = x + 3 \text{ (Original function)}$$

$$y = x + 3 \text{ (Replace } f(x) \text{ by } y)$$

$$x = y + 3 \text{ (switch } x \text{ and } y \text{ to obtain inverse)}$$

$$y = x - 3 \text{ (expressing the equation in terms of } y)$$

$$f^{-1}(x) = x - 3 \text{ (inverse function)}$$

Let us now sketch the graphs for the function $f(x) = x + 3$ and inverse function $f^{-1}(x) = x - 3$

One way to sketch the graph is to find the intercepts.

Graph of function $f(x) = x + 3$

Step 1: we first replace $f(x)$ by y and then find the y -intercept, let $x = 0$.

$$y = x + 3$$

$$y = 0 + 3$$

$$y = 3$$

The line passes through $(0, 3)$, so the y -intercept is 3

To find the x -intercept, let $y = 0$

$$0 = x + 3$$

$$x = -3$$

The line passes through $(-3, 0)$, so the x -intercept is -3 .

Step 2: We plot $(0, 3)$ and $(-3, 0)$ and then we draw a straight line through them.

Graph of inverse function $f^{-1}(x) = x - 3$

Step 1: replace $f^{-1}(x)$ by y

$$y = x - 3$$

To find the y -intercept, let $x = 0$.

$$y = 0 - 3$$

$$y = -3$$

The line passes through $(0, -3)$, so the y -intercept is -3

To the x -intercept, let $y = 0$

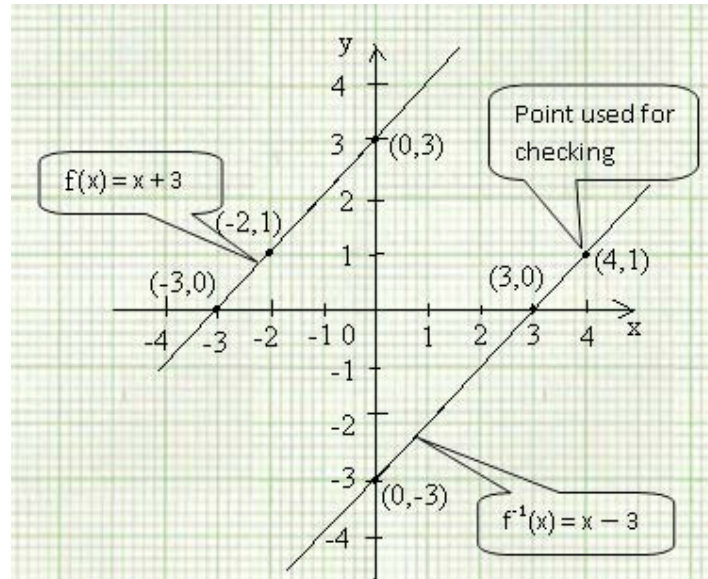
$$y = x - 3$$

$$0 = x - 3$$

$$x = 3$$

The line passes through $(3, 0)$, so the x-intercept is 3.

Step 2: Plot $(0, -3)$ and $(3, 0)$. Draw a straight line through them.



Feedback

We hope that after comparing your answers with the model answers provided above, you might have got all the answers correct. Congratulations! If not then I suggest you try the questions you got wrong again. You may be required to read through the section again in order to gain more understanding.

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Unit 3

Graphs of Polynomials

Introduction

In unit 1 you learnt about approximations and scientific notation. The knowledge you gained in approximation is vital. You will use it to estimate values as you study this unit. In unit 2 you studied relations and function. You dealt with the relationship between variables. This knowledge will be useful as you study the relationship between variables in graphs. As already stated, these units are very important in your this course and there are several reasons for this. One of the reasons is that you will be examined at the end of your course together with other units you learned in grade 10 and those you will learn in grade 12 in order for you to qualify for the General Certificate of Education. One other reason is that mathematics is very applicable in our everyday life. For example, commercial and social arithmetic knowledge, to mention but a few, is used almost daily in our lives unconsciously. You learn how to calculate the utility bills such as electricity, water and many other such examples.

This unit has three topics. The content of some of these topics are new to you while other content you covered in your Junior Certificate Program. For instance, in your junior program, you learnt about the XOY-plane. You learnt about the equations in two variables. You went further to learn how to draw the graph of a straight line. This will be explained in detail and new knowledge will be added to what you already know.

In each topic, you will be required to do some activities and a topic exercise. You are encouraged to answer all the questions in the activities and topic exercises.

After studying this unit, you will be required to do tutor marked assignment 1(TMA 1) which you will send to your tutor for marking. This is your first tutor marked assignment of this course. This TMA is based on the first three units covered so far. You will, therefore, also need the knowledge from the other two units to answer the questions in this TMA.

After studying this unit, you will be expected to achieve the outcomes listed below.

Upon completion of this unit you will be able to:

Outcomes



- *Draw and interpret* graphs of straight lines.
- *Draw and interpret* graphs of quadratic and cubic functions.
- *Calculate* the gradient of a curve at a given point.
- *Draw graphs* using out of real life situations.
- *Estimate* the area under the curve.

Time Frame



It is estimated that to complete studying this unit you will need between 13 to 14 hours. Do not worry if you take longer in finishing this unit. Remember that we all have different abilities and hence we study at different paces.

You are encouraged to spend about 2 hours answering each topic exercise in this unit. Since there are three topic exercises in this unit, this means that you will spend about 6 hours on these exercises.

You will be required to spend about 3 hours on the tutor marked assignment.

The total hours for completing the unit will thus be between 22 and 23 hours.

Learning Resources



In order to study this unit with minimal difficulties you will need the following materials:

- Ruler
- Pencil
- Graph Paper
- Eraser

Teaching and Learning Approaches



As you study this unit, the emphasis is on your understanding the principles of drawing a graph and interpretations. This will be achieved by working out the procedures of drawing graphs and the actual drawing of graphs. You will be able to master the skills as you practice the skills in working out the activities and exercises as an individual alone or in a group with others studying the same unit. You will also have an opportunity to discuss your work with your tutor/tutors at your nearest learning centre.

This unit is interactive in approach. As you study, you will find activities with blank spaces left in the unit. These spaces are meant for you to use as you interact with the materials. You need to use them to write down some notes as you study or even work out certain questions in the activities given to you. As you study the topics, it is necessary for you to assess yourself. To assess your knowledge and skills being acquired you have to work out the activities and the exercises and mark your own work using the feedback provided immediately. It is expected of you not to go through the feedback before doing the activity.

Terminology



Abscissa:	The x- value in a given coordinate.
Axis::	This is a line.
Coordinate:	A set of a pair of numbers used to define a position with respect to the XOY- plane.
Domain:	A set containing initial input values of a function.
Expression:	Refers to a mathematical statement such as $bx + dx$ or $3x + 4y - 2a$.
Function:	It describes a relationship between two or more variables.

Gradient:	This is referred to as the slope or the inclination of a line to the horizontal line.
Intercept:	The point at which the curve or line cuts the axes. The X- intercept is on the X-axis and the Y intercept is on the Y-axis.
Ordinate:	The y-value in the coordinate point.
Parabola:	This is the shape of the curve drawn of the quadratic function.
Polynomial:	An algebraic expression or equation in which one term is raised it to a power, a non-negative integer.
Tangent Line:	A straight line drawn touching the curve at one point only.
Transition:	Is a process of moving from one area to another.
Variable:	A letter used in a term of an expression or equation such as; $2x$ or $-5y$ where x and y are in this case variables.
Vertex:	The point of intersection of the parabola and the axis of symmetry.

Topic 1 Graphs of Linear Functions

As earlier stated, this is the first topic in which you will learn about linear functions. You will learn how to draw the graphs of linear functions. You were introduced to linear functions when you learnt about equations in two variables. These produced straight line graphs when drawn on the XOY-plane. You learnt how to make the table of values for x and y variables by calculating the values of the Y-variable given in a function or the equation. You also learnt how to plot the points as coordinates on the plane and finally draw the graph. This time you will learn this topic in detail. You will learn how to find or identify the x - and y -intercepts of the graph. You will further learn how to interpret the linear graphs given a graph

situation. As we already said, after studying the topic you will be required to answer self-assessment questions at the end of the topic. You are encouraged not to go through the feedback before doing the topic exercises.

Linear functions are used in our everyday life situations to determine the relationship between two or more variables. For instance, a metal expanding due to heat and contracting due to the coldness determines the relationship between the length and the temperature. In your junior science, you learnt about the expansion of metals. You learnt that when metals are heated, they expand and when cooled, they contract. This means that when a metal is heated, it increases in its length. This increase in length is as a result of the temperature increase. When you check railway lines, you will notice that the metals are laid down in such a way that there is a gap at the ends of the metals. This is to ensure that when metals are expanded, there is room for increase without warping or interrupting the railway system. In this situation, the expansion in length of the metal depends on the temperature. In such a case, the length is a function of the temperature. This and many more situations in life are dealt with under linear functions.



In this first topic of the unit, we will address the first of the five unit outcomes, which is to:

- Draw and interpret the graphs of linear functions.

For us to achieve this outcome, you should be able to achieve the following objectives:



- Find the x- and y- intercepts of the straight lines.
- Find the coordinates of the linear graph.
- Draw the graph of the linear function.
- Interpret the given linear graph.

Linear Functions

As earlier stated, linear functions is not a new topic for you. You covered this topic under equations in two variables in your Junior Program. We are now continuing from where you ended.

Linear functions or linear equations are sometimes expressed in the standard form of $y = mx + c$, where m is the gradient and c is the intercept on the y – axis.

For the sake of our study, we will use this expression of the linear function. The intercept has already been defined in the terminology section as the point where the line or graph cuts the y axis or x axis.

For example, in the diagram (Figure 1) below, the x intercept is -2 and the y intercept is $+3$. Study the diagram to see what we are referring to. I am sure that negativity or positivity of a number was covered under the set of integers which you learnt in the number and numeration topic of your junior program and also in grade 10.

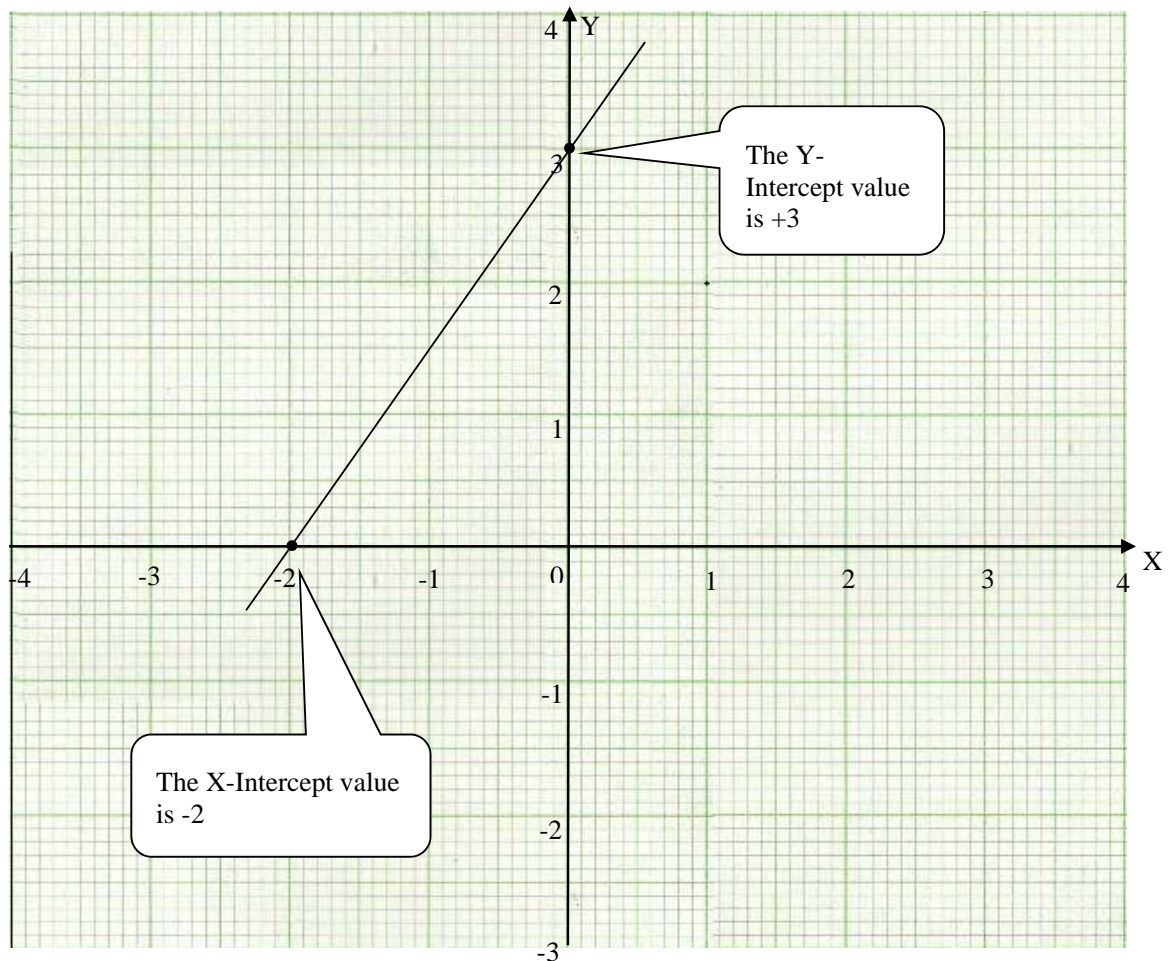


Figure 1: The diagram shows x and y intercepts.

From the graph above, we can pick out the intercepts as illustrated above. On the Y-intercept, the value of x - is zero (0) and on the X-intercept, the value of y - is zero (0).

In short, the point coordinate of the Y-intercept is written as $(0, y)$ and the point coordinate of the X-intercept is written as $(x, 0)$. Remember that you have already dealt with writing coordinates in your junior program as stated above.

For the graph above, the Y-intercept point is $(0, 3)$ and the X-intercept point is $(-2, 0)$

The following example is to help you to draw a graph of a linear function. It will also help you to revise writing coordinates, make the table of values and identify the intercepts.

Example 1

Draw the graph of the function $f(x) = 2x + 2$

Solution

In drawing such graphs, you should first have a table of values for the x and y values. You should note that at times you are not given the x -values. In such situations, you have to choose values of x to suit your own convenience.

For the discussion purpose, you take the values of x from -2 to 2 . This means that you take values of x as follows: $-2, -1, 0, 1, 2$. Remember that from your junior work, you learned that every graph needs a scale. This guides you on what values to include on the axis or on both axes. In this example, 2 cm of the graph paper will represent 1 unit on the x -axis and 2 cm on the graph will represent 2 units on the y -axis.

With the above information in mind, you substitute each value of x in the function above to get the values of y as shown in the table below. In your grade 10 work on literal equations and formulae, you learnt how to substitute values for the variables and finally evaluating the expression given. This is the knowledge required here to make the table of values.

Table 1: Showing values of x and y for the function $y = 2x + 2$.

X	-2	-1	0	1	2
y	-2	0	2	4	6

These values form pairs of coordinates in the form (x, y) .
Therefore the points are $(-2, -2), (-1, 0), (0, 2), (1, 4),$ and $(2, 6)$.

These points are to be plotted on the XOY-plane as shown below:

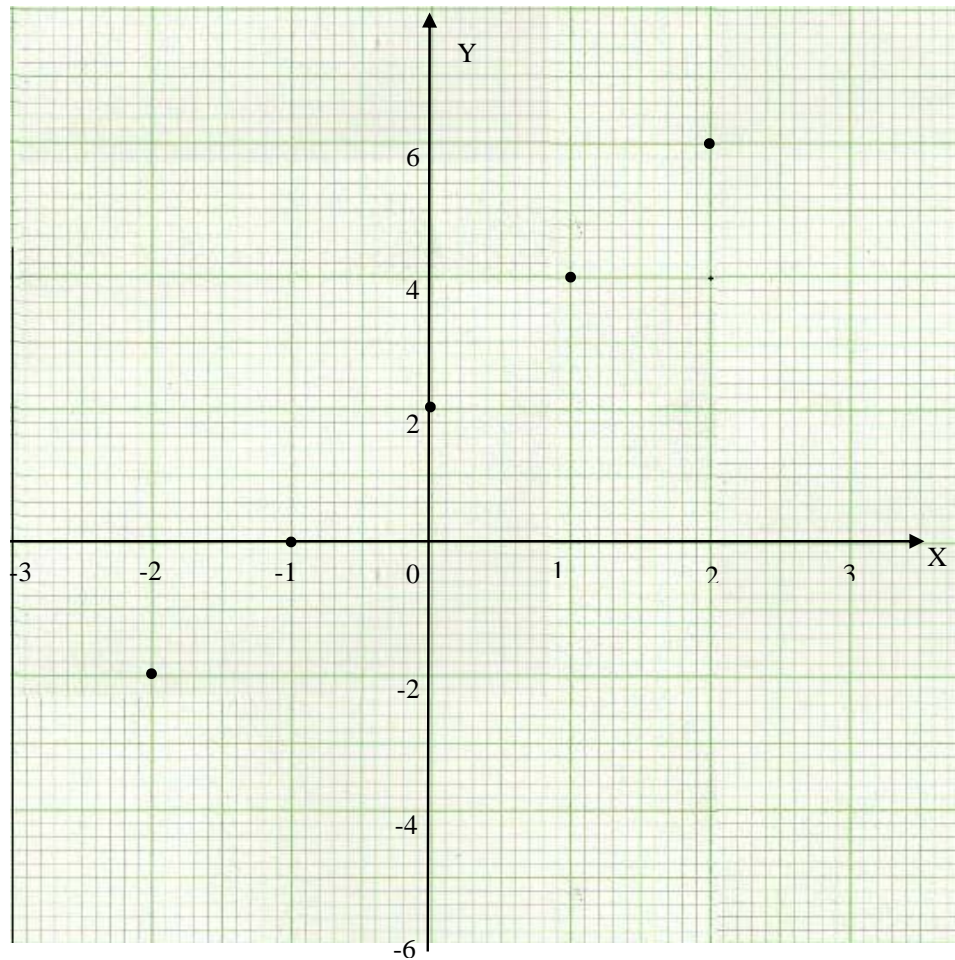


Figure 2: Plotted points

The figure above illustrates the plotted points on the graph. These points were first calculated using the equation or rather the function, $y = 2x + 2$ as we are addressing it here.

When these points are joined together by a straight line, the line is referred to as the Straight Line Graph as shown below.

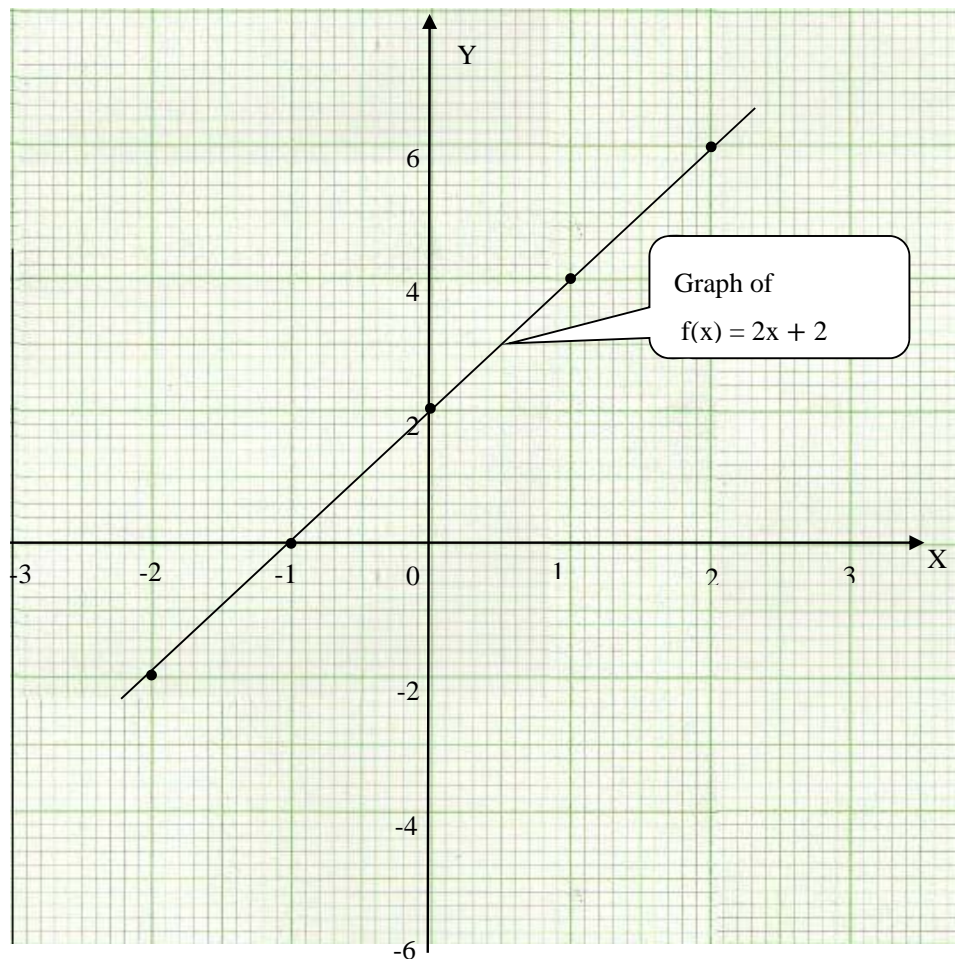


Figure 3: The graph of $y = 2x + 2$.

The first thing we did was to plot the points as shown in figure 2. Thereafter, using the points as a guide you draw a line to pass through all the point plotted on the graph. Mind you, this graph can run as far as it can go on both ends. The guide of where to start and stop is the points on the graph. If we had further point up or down on the plane from the same function, we would have gone as far as these points. The nature of such graphs is that they pass through the points plotted.



The graph of a linear function, when defined on a set of real numbers, forms a straight line.

Now let us consider another example. This example will help you appreciate the skill of drawing a graph. It is similar to the one above in that the graph will align in the same direction but the y-intercept will be different.

Example 2

Draw the graph of $y = 3x - 2$ for the domain $-2 \leq x \leq 5$, where $x \in \mathbb{R}$.

Solution

To draw such a graph, first you put up a table of the values of x and their corresponding values of y for the domain given.

That is $x = \{-2, -1, 0, 1, 2, 3, 4, 5\}$.

Now let's use these numbers to find the values of y in the table

X	-2	-1	0	1	2	3	4	5
$3x - 2$								

together by completing the table below.

Table 2: Showing only the x values.

Remember that the values of y are calculated by substituting the values of x in the function: $y = 3x - 2$.

Substituting the values of x in the linear equation $y = 3x - 2$, we obtain the corresponding values of y .

When $x = -2$, we have

$$y = 3x - 2$$

$$y = 3(-2) - 2$$

$$y = -6 - 2$$

$$y = -8$$

-6 was found by multiplying 3 by -2 as shown

When $x = -1$, $y = 3x - 2$

$$y = 3(-1) - 2$$

$$y = -3 - 2$$

$$y = -5$$

-3 was got by multiplying 3 by -1

When $x = 0$, $y = 3 - 2$

$$y = 3(0) - 2$$

$$y = 0 - 2$$

$$y = -2$$

When $x = 1$, $y = 3 - 2$

$$y = 3(1) - 2$$

$$y = 3 - 2$$

$$y = 1$$

When $x = 2$, $y = 3 - 2$

$$y = 3(2) - 2$$

$$y = 6 - 2$$

$$y = 4$$

When $x = 4$, $y = 3 - 2$

$$y = 3(4) - 2$$

$$y = 12 - 2$$

$$y = 10$$

Finally when $x = 5$, $y = 3 - 2$

$$y = 3(5) - 2$$

$$y = 15 - 2$$

$$y = 13.$$

After all these calculations, you then write the values of y in the table as shown below:

X	-2	-1	0	1	2	3	4	5
Y	-8	-5	-2	1	4	7	10	13

Table 3: Showing a completed table of values of x and y .

You can now plot these points on the coordinate plane. You use the scale: 1 cm to represent 1 unit on the X-axis and 1 cm to represent 2 units on the vertical axis.

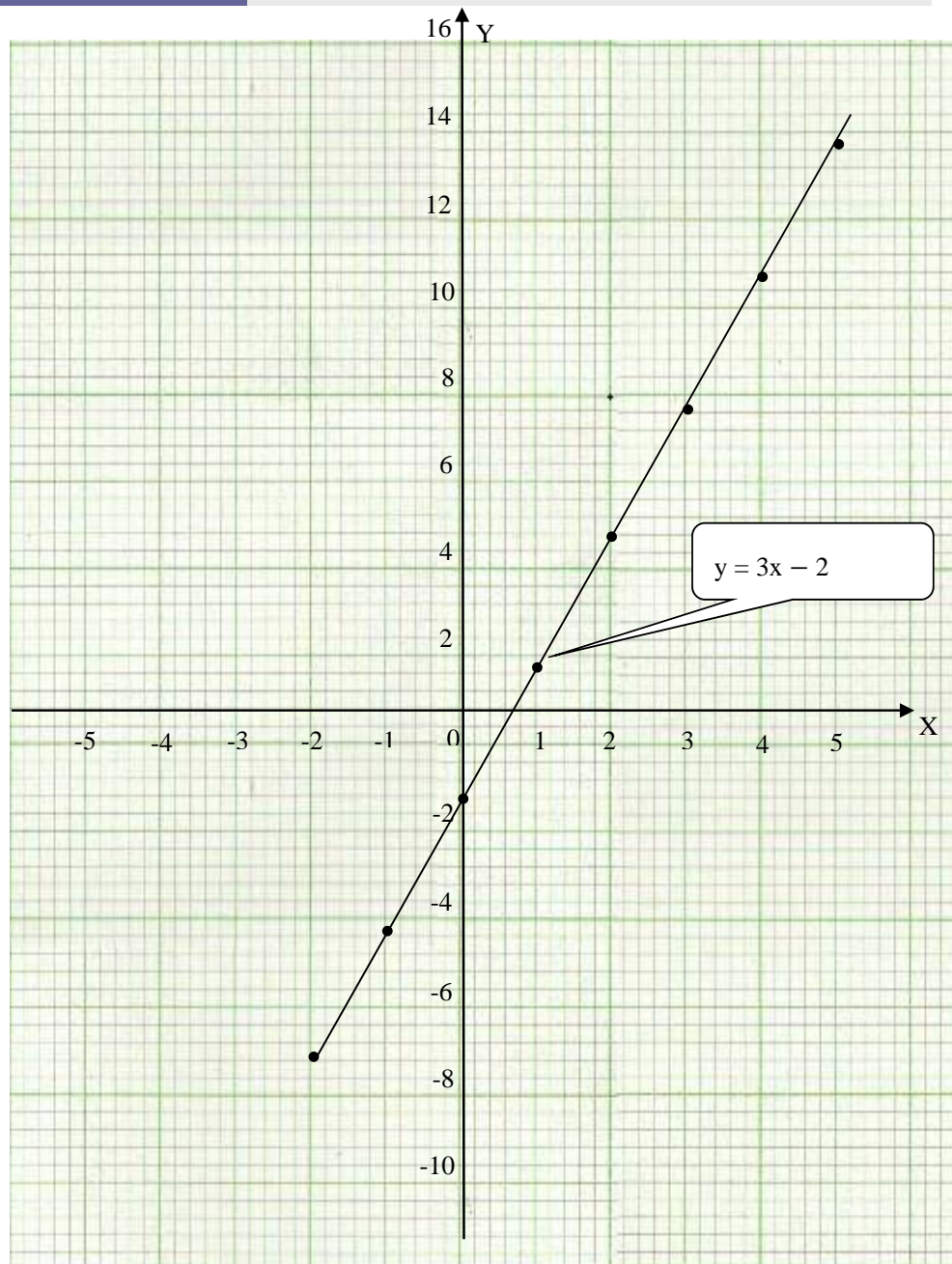


Figure 4: The graph of $y = 3x - 2$ for $-2 \leq x \leq 5$.

When the corresponding points are plotted and joined, we have the straight line as in the diagram above. As you can see from the graph, the line passes through -2 as its y-intercept. Apart from that, the graph aligns in the same direction as the first graph drawn of $y = 2x + 2$.



Now here is an activity for you to do. This activity is very similar to the two above which we have just done. It will help you master the art of drawing graphs of linear functions. This is similar to what you did but the graph may align differently than above.

Activity 1

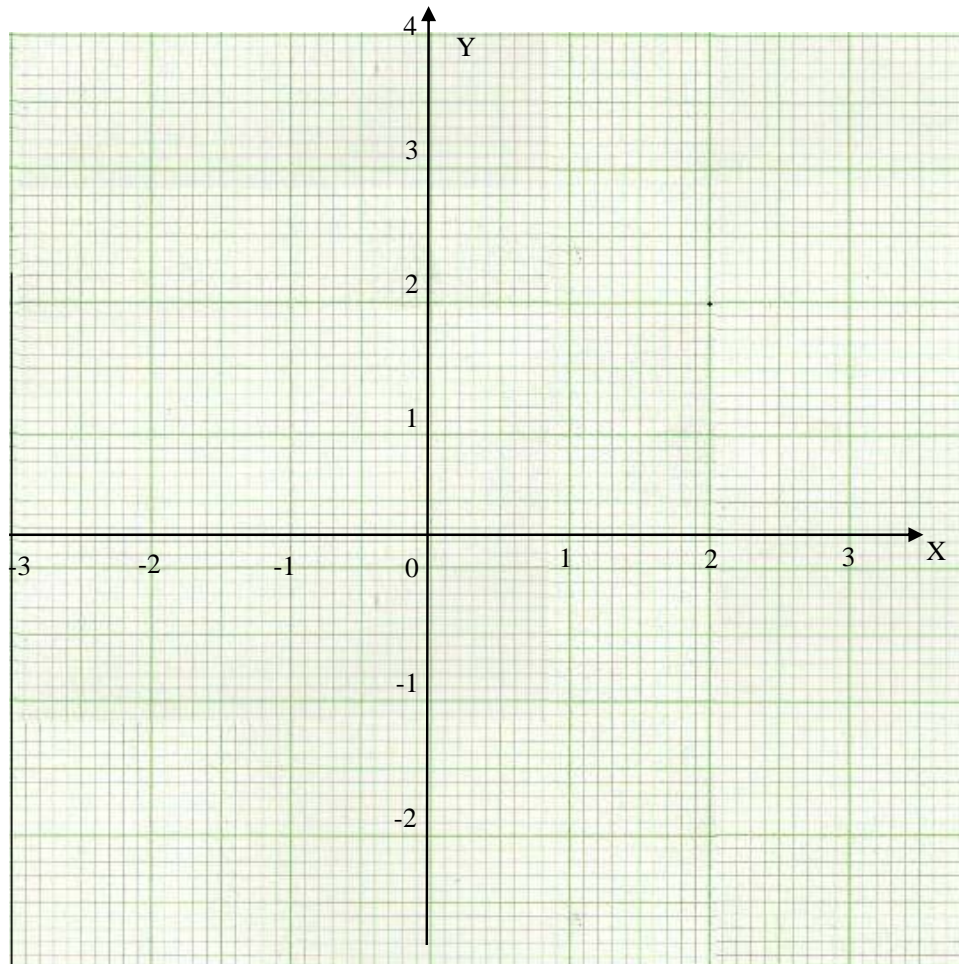
Consider the graph of $y = 2 - x$, for the domain $-2 \leq x \leq 3$. Draw the graph on a graph paper and show your calculations in the space below.

Complete the table of values below.

x	-2	-1	0	1	2	3
$2 - x$						

Table 4: showing the x values for $y = 2 - x$

Use the following grid for your graph



The feedback is to help check your work. You should only compare with the feedback after you have worked on your answer.

It is assumed that you have applied the knowledge you have acquired from the two examples above. You should have noticed that the work was very easy as it was the same as the examples. You can compare your work with the work in the feedback below and make corrections where necessary.

You should have first found the values of y for the given values of x .

You needed to complete the table of values for $y = 2 - x$ below.

X	-2	-1	0	1	2	3
$2 - x$						

Substituting the values of x in the equation $y = 2 - x$, you obtain the corresponding values of y .

When $x = -2$, $y = 2 - x$

$$y = 2 - (-2)$$

$$y = 2 + 2$$

$$y = 4$$

From Integers: $-(-2) = 2$

When $x = -1$, $y = 2 - x$

$$y = 2 - (-1)$$

$$y = 2 + 1$$

$$y = 3$$

When $x = 0$

$$y = 2 - x$$

$$y = 2 - 0$$

$$y = 2$$

When $x = 1$, $y = 2 - x$

$$y = 2 - 1,$$

$$y = 1$$

When $x = 2$,

$$y = 2 - 2$$

$$y = 0$$

When $x = 3$ $y = 2 - x$

$$y = 2 - 3$$

$$y = -1$$

Therefore the values of x and corresponding values of y obtained can be written as shown in the table shown below.

X	-2	-1	0	1	2	3
Y	4	3	2	1	0	-1

Table 5: Completed Table of values for $y = 2 - x$

You now plot these points on the coordinate plane. When you use the scale of 1 cm to represent 1 unit on each axis, you have the following graph.

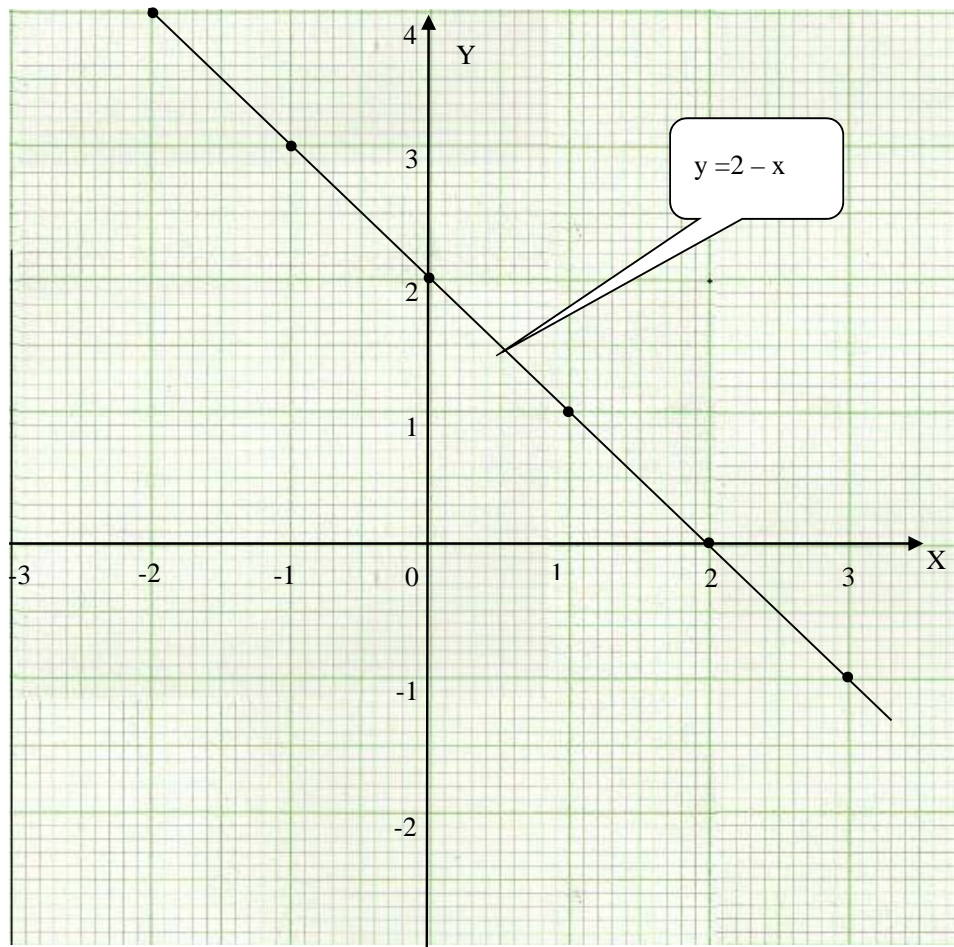


Figure 5: The graph of $y = 2 - x$

The graph of a straight line can be drawn on graph paper taking an appropriate scale. As you noticed in your work, the graph has changed or has a different alignment. It is very different from the other two above.



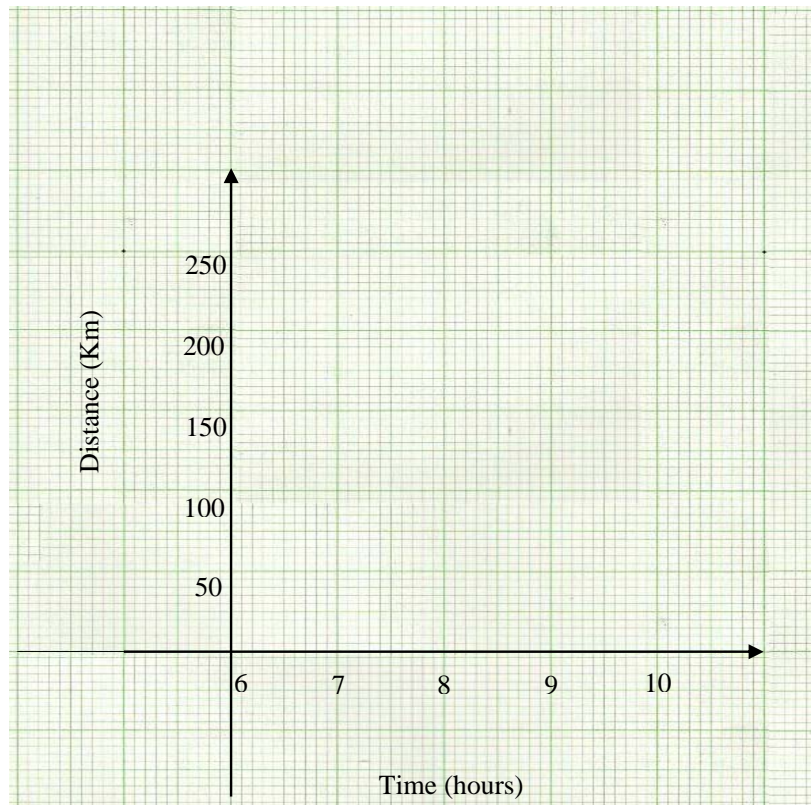
Let us consider another activity. This activity is to help you learn to understand and appreciate mathematics from our everyday life situation. Graphs are not done for just the sake of it. In this activity you see the use of mathematics in daily life.

Activity 2

Mr. Mbewe cycled from Kasempa to Solwezi, a distance of about 185 km on his Motorbike at a speed of 60 km/h. He started off from Kasempa at 06 00 hrs. Using a scale of 2 cm to represent 1 hr on the horizontal axis and 2 cm to represent 20 km on the vertical axis;

- Draw the graph of Mr Mbewe' s Journey
- Using the graph, estimate the time he arrives in Solwezi.

Use the following graph grid to do your work.





In this feedback pay particular attention to the use of variables. The two that have been used here are time and distance covered in that time. You should check it through after you have done the activity.

It is hoped that having done the examples above you found this activity very interesting. You may check your work with the feedback given below.

You should have made the table of values using the information about the speed of the motorbike. The table of values will be as shown below:

<i>Time (h)</i>	<i>6 00</i>	<i>7 00</i>	<i>8 00</i>	<i>9 00</i>	<i>10 00</i>
<i>Dist.(km)</i>	<i>0</i>	<i>60</i>	<i>120</i>	<i>180</i>	<i>240</i>

Table 6: Showing time-Distance values.

From the above table the graph is then drawn as shown below:

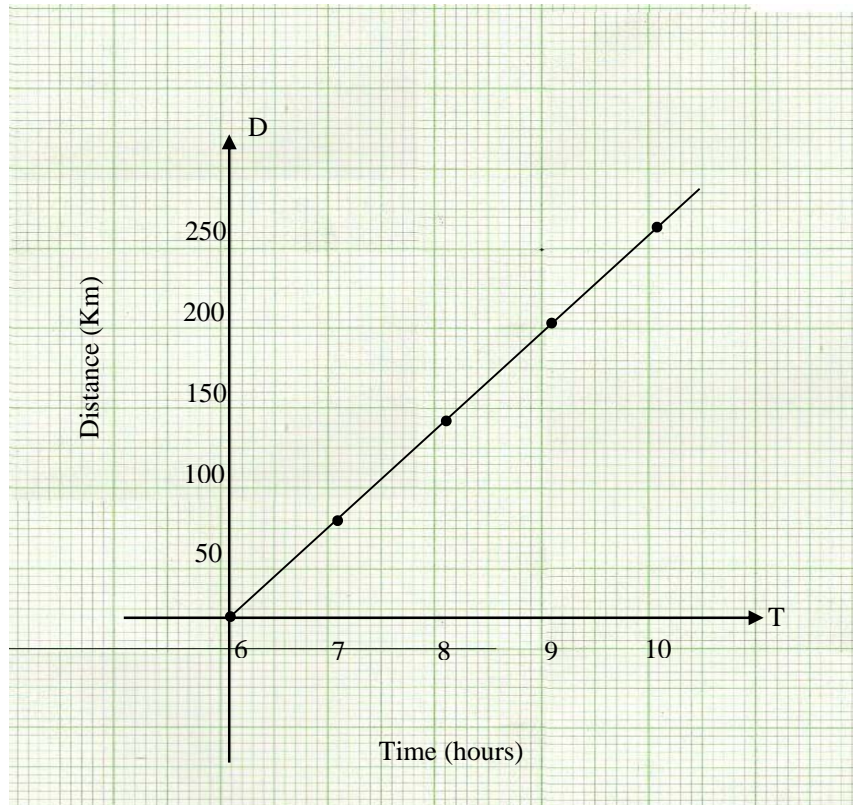


Figure 6: Distance-Time Graph of Mr. Mbewe 's Journey

You have so far learnt about finding the x- and y- intercepts and drawing of the graph. Now you have to use this knowledge in studying the following part of the topic. As you handle interpretation of the linear graphs, you will be required to use the skills and knowledge that you have so far learnt.

Interpretation of the Linear Graphs

As it has been earlier stated above, you are now going to learn how to interpret graphs. In this part of the topic, you are expected to use your knowledge so far acquired to deal with this section. In your junior program, you learnt how to find the gradient of a straight line. You also learnt how to write an

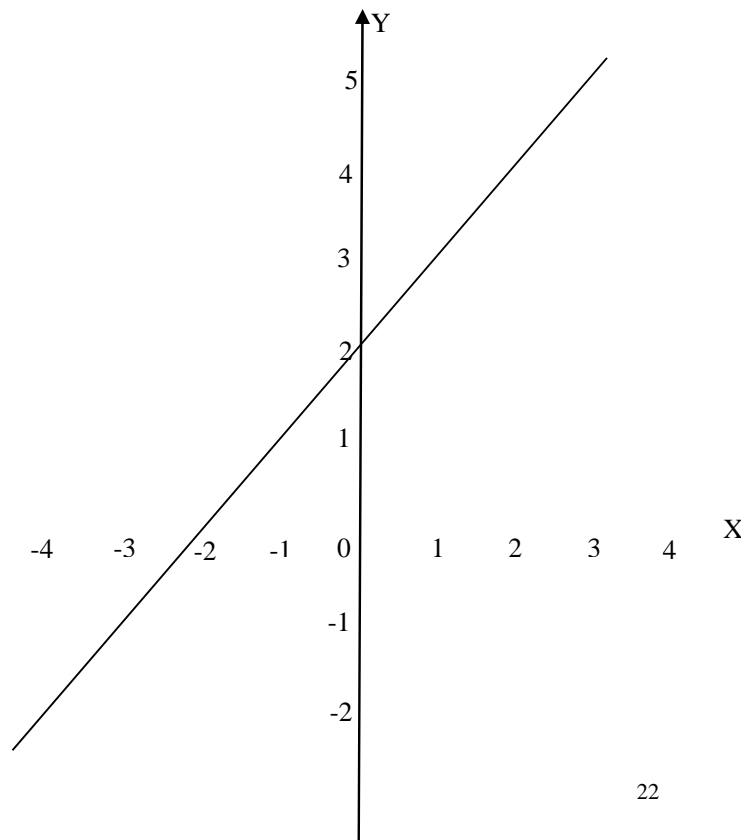
equation of the graph given. This is the knowledge you will require.

Graphs in life help us understand quantitative information. This information is usually in the form of a line representing such a given function. This helps one visualise comparisons or changes.

You learnt to draw a graph given a function. You learnt to identify the intercepts and the gradient given the slope-intercept form of the equation of the function. The following example is on interpretation of graphs. It will help you just understand how to calculate the gradient of a given line and how to write down the equation of the function.

Example 4

Using the diagram below, calculate the gradient of the graph in the diagram and write down the equation of the same graph.



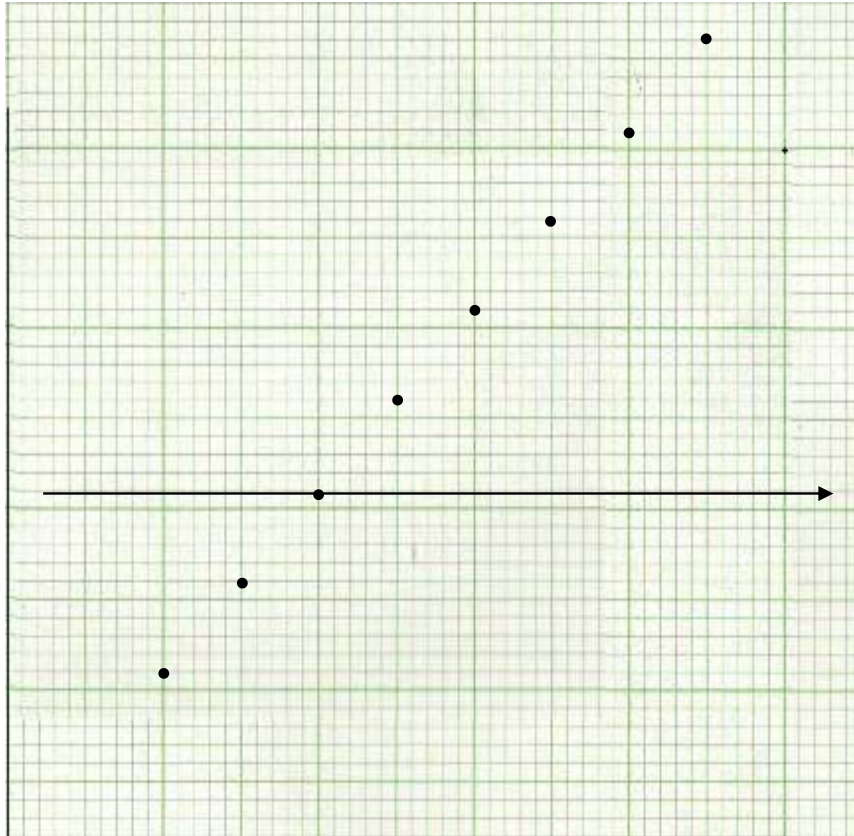


Figure 6: graph of a linear function.

Solution

In your application of the knowledge and skills of drawing the straight line graph, you learnt about the gradient and intercepts. You also learnt about the equation of a straight line. You are now expected to use that knowledge to help you arrive at the answer.

You first have to find the gradient using the gradient formula. Remember that you learnt how to use this formula in your junior program when you dealt with the gradient and the straight line. However, as a way of reminding you of your previous work, the formula will be written in full and you are to use it in the process of calculating the gradient.

You first pick out any two points on the line. The two points are for your convenience in ensuring that you find the gradient. Remember that any two points picked on the same line should be able to give you the same answer. In this case let us pick; $(-3, -1)$ and $(1, 3)$. Please note that you may pick any other two points of your convenience from the line. These two points will help us understand the relationship between the variable y and the variable x .

Firstly, we have to find the gradient of the line. Since it is a line, the gradient of this line is the same at any point on the line.

By use of the gradient formula which is:

$m = \frac{Y_2 - Y_1}{X_2 - X_1}$ where m is the gradient of the line, Y_1 and X_1 are from one point while Y_2 and X_2 are also from another point.

Let us make $(-3, -1)$ be the point (X_2, Y_2) and $(1, 3)$ be the point (X_1, Y_1) ;

Therefore substituting in the formula, $m = \frac{Y_2 - Y_1}{X_2 - X_1}$, you have:

$$m = \frac{-1 - 3}{-3 - 1} = \frac{-4}{-4} = 1$$

Therefore, the gradient of this line is 1. This gradient also shows that the line is gently steep.

The value of the gradient shows the steepness of a line. Smaller values indicate that the line is gently steeping while large values indicate that the line is very steep. In short as the value increases, the gradient also gets steeper. The negative or positive value of the gradient shows the alignment. A negative value of gradient shows that the line is from top left going downwards to the right and for the positive value, the line is from top right dropping downwards to the left.

To now write down the equation of the graph, you have to see the relationship between the variables. We say that y is a function of x , this is shown as: function y relates the variable y to the variable x in some way defined only by a rule,

To find the relationship, we use the same gradient formula:

$$m = \frac{Y_2 - Y_1}{X_2 - X_1}$$

We pick only one point from the two points given above and the gradient value and substitute them in the formula. So we pick $(1, 3)$ as the point (x, y) . Whenever we pick two points, we calculate the gradient. In formatting the equation of the line, a point is used. This point is substituted in the gradient formula to formulate the equation as shown in the working below:

$$m = \frac{Y_2 - Y_1}{X_2 - X_1}$$

$$1 = \frac{Y - 3}{X - 1}$$

$$1(x - 1) = y - 3$$

$$x - 1 = y - 3$$

$$y = x - 1 + 3$$

$$y = x + 2$$

Cross multiply here to get rid of the fraction.

Make y the subject of the formula from this stage. In making y the subject, we are solving for the value of y . We make our equation.

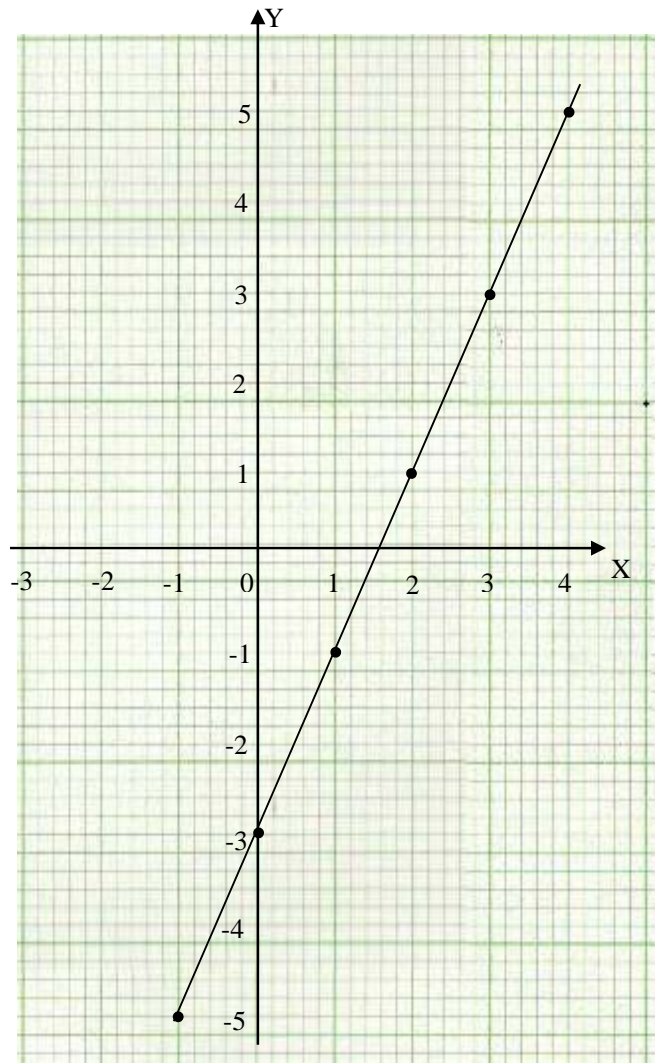
Therefore, the relation between y and x is that you add 2 to the value of x to get the value of y .



The following activity is similar to the work done above. The graph itself is passing through different values but the process is the same. It will help you master the skill of finding the gradient and finally writing down the equation of the line represented by the graph given.

Activity 3

Study the diagram given below.



- (a) Find the gradient of the line.
(b) Write down the equation of the line.

Write the answers below.



You are now looking at the interpretation of a given graphs. You are calculating gradient and finding the equation of the line or curve. You should check it through after you have done the activity.

It is hoped that you found this activity very easy to work through as it was similar to the earlier one in terms of the working process. The knowledge from the example should have made you work through the activity with ease. Compare your working with the working in this feedback.

The first thing you should have done was to pick two points from the line to be used in this situation. Remember that whatever points you pick, they will give you the same gradient. Let us pick the following points: (-1, -5) and (4, 5).

The gradient: $m = \frac{y_2 - y_1}{x_2 - x_1}$. Let us take $x_1 = -1$, $x_2 = 4$, $y_1 = -5$ and $y_2 = 5$.

$$m = \frac{5 - (-5)}{4 - (-1)} = \frac{5 + 5}{4 + 1} = \frac{10}{5} = 2$$

Therefore, the gradient is 2.

For the equation of the graph, you should have used any one point between the two that you picked. Now using any one of the two that we have picked here;

(-1, -5)

The equation is given by:

$$M = \frac{y - y_1}{x - x_1}$$

$$M = \frac{y - (-5)}{x - (-1)} = \frac{y + 5}{x + 1}. \text{ We know that } m = 2$$

$$2 = \frac{y + 5}{x + 1}$$

Cross multiply here

$$2(x + 1) = y + 5$$

Expand the brackets

$$2x + 2 = y + 5$$

$$y = 2x + 2 - 5$$

$$y = 2x - 3$$

Make y the subject of the formula

Therefore, the equation is $y = 2x - 3$

This means that the value of y is equal to multiply the x value by 2 and then subtract 3 from the product.

Topic Summary



In this topic on linear functions, you have learned how to find the x - and y - intercepts of a given graph. You also learned to find the gradient of a straight line graph. You went on to learn how to draw the graph of a straight line graph and the interpretation of the same.

We also noted that:

- The equation of the graph is written in form of $y = mx + c$, where m is the gradient and c is the y -intercept.
- The gradient of any straight line graph is the same at any point on the line itself.
- The gradient can be calculated using any two points on the line.

You were also encouraged to answer all the questions in the activities provided in this topic. The activities were intended to help you assess how well you understood the content of the various sub-sections of the topic. If you did not do very well in the activity, this means that you need to go over the material again.

In the next topic we will discuss the graphs of quadratic and cubic functions.

Now you have learnt how to draw the graphs of linear functions, finding the gradient and even interpreting the graphs. In order to know how much you have learnt you should do the End of Topic Exercise 1 which you will find at the end on the unit in the assignment section. As already stated in the other activities, once you have completed the exercise, check your answers by comparing them with the feedback provided.

Topic 2 Graphs of Quadratic and Cubic function

As earlier stated this is the second topic. You have so far learnt how to identify the intercepts, draw the straight line graphs and interpret the graphs. This knowledge would be used in this part of the unit. You will be expected to identify the intercept of the graph and then move on to drawing the graphs. You will learn how to draw the graphs of quadratic and cubic functions.

When you dealt with factorisation of quadratic expressions both in your junior program and in Grade 10 works, you were introduced to this function. This time you will learn how to draw the quadratic and cubic function graphs. You will also learn how to find or identify the x- and y-intercepts of the graph. You will further learn how to interpret the quadratic and cubic graphs given a graph situation. In the interpretation, you are expected to identify the maximum and minimum point values.

After studying the topic you will be required to answer self-assessment questions at the end of the topic. You will find this end of topic exercise in the Assignment Section at the end of the unit. As in the first topic and in all units of this course, you are encouraged not to go through the feedback before doing the end topic exercise.

It is important for you to understand that these graphs are very useful in our everyday living. For instance, when constructing a new highway, civil engineers consider the distance at which a driver will be able to see the upcoming road conditions. This means that when they are connecting two stretches of road where the slopes of the land are different, they design a transition curve. This curve which provides the smoothest transition is a parabola.



In this second topic of the unit, we will address the next three of the five unit outcomes. In this part, we will address drawing and interpreting the graphs of quadratic and cubic functions. We will also address calculation of the gradient of a given curve at a point and finally we will address drawing of graphs using out of real life situations.

To achieve these, the following are the objectives to be covered by the end of the topic:



- *Find* the x- and y- intercepts of the quadratic and cubic graphs.
- *Identify* the axis of symmetry of the quadratic graph.
- *Draw* the graph of the quadratic and cubic function.
- *Determine* the minimum or maximum of the quadratic graph.

Quadratic functions

From your junior program, you dealt with factorisation of quadratic expression of the form, $ax^2 + bx + c$. From your work in Grade 10, a quadratic expression was defined as a mathematics statement in one variable and in which the variable has the highest power of 2 as shown above in the first sentence. In the equation, ax^2 shows the power of 2.

A quadratic expression becomes an equation when an equal sign is introduced. When the expression above is written as $ax^2 + bx + c = 0$, it becomes an equation.

When the equation is written as $f(x)$, it is now referred to as quadratic function, which is in the form $f(x) = ax^2 + bx + c$, where a , b and c are real numbers and $a \neq 0$. The f in $f(x)$ refers to the word function. The value c is called the constant of the equation or the y-intercept value of the graph. When you learnt graphs of linear functions, you learnt about the y-intercept which is the point at which the curve cuts the y-axis. It is the same knowledge that is been explained here. When this function is drawn on the graph, its graph is referred to as a Parabola. It takes up two shapes. This graph is symmetrical about a vertical line called the axis of symmetry. When a shape has a line (axis) of symmetry, it means that a shape can be folded along that line and it fits its own shape exactly. The two halves of the shape when folded along the line (axis) fit the outline of each other exactly. You will learn this knowledge of symmetry in Unit 8 on Lines of

symmetry. When the coefficient of x^2 (a) is negative, the graph is in cap form and has a maximum point. When the coefficient of the x^2 (a) is positive, the graph is in cup or U shape and has a minimum point. In grade 10 work you learnt on Processes of Algebra, you learnt that an expression of a nature such as $5x$ is called a term. In the term, the number 5 is called the coefficient of the term in x and the x as you may already know is a variable. When a term is written as $-y$ or y , it still has the two parts. It has a coefficient and a variable. In these cases shown as $-y$ or y , the coefficient is -1 or 1 . Note that any term without a variable such as 4 , is called a constant. Be mindful that at times in general statements in mathematics, the coefficient is written as a letter as in the following: $ax^2 + bx + c$. You will notice that a and b are coefficients, c is the constant and x is the variable. Now with that reminder, let us continue.

The following example is to help you understand the concepts put forward above. This graph has a cup or U shape form since the coefficient of x^2 is positive. You should note that if this was negative, the shape would be in CAP form or \cap shape. It will help you understand the axis of symmetry and the vertex, which in most cases is referred to as either the maximum or minimum point.

Example 1

From the graph of the quadratic function $f(x) = x^2 + 2x - 8$ given below, determine the x - and y - intercept, the axis of symmetry and the vertex of the function.

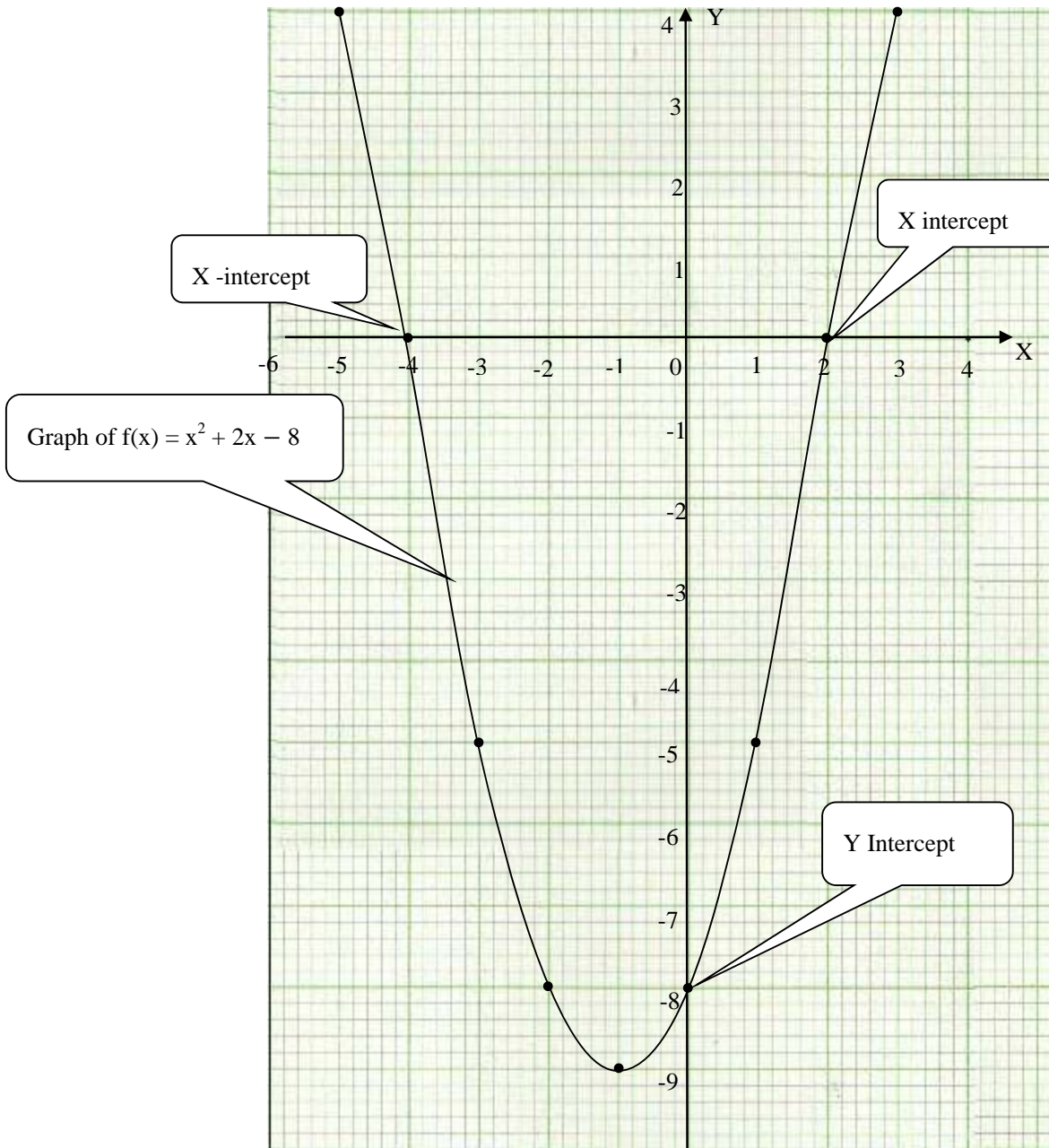


Figure 7: The graph of $y = x^2 + 2x - 8$

Solution

From topic 1, we know that the intercept is the point of intersection of the curve with the axes.

Remember that the y-values at the point of intersection with the X-axis is always zero (0).

On the X-axis, the curve cuts or intersects the axis at two points given as $(-4, 0)$ and $(2, 0)$. Please take note of these points on the graph above and confirm them. You will notice that $(-4, 0)$ is on the X-axis on the negative part of the line and that $(2, 0)$ is on the positive part of the line. You will notice also that the curve comes down from both left and right sides and at point $(-1, -9)$, the curve turns upwards. If you use the left side, it means the curve turns upwards to the right. If you use the right side, it means again that the curve turns upwards to the left. Notice also that the values of y decreases as the curve comes downwards either from left or from right. When it reaches the point $(-1, -9)$, it turns back upwards. This point, at which it turns upwards, is called the minimum point. If you take the value of y in the range of values of y in the table for the graph, you will notice that this is the lowest number (value) or minimum value. Using this knowledge, the value of -9 is the minimum value of y for this graph.

Now let us consider the points of the x-axis. You have noticed that the y-values are zero in both points. This means that if the equation is written as: $y = ax^2 + bx + c$, we are calculating for the y value. Now when y is given as zero, we substitute in the equation, it becomes: $0 = ax^2 + bx + c$. When this happens, we are now solving for the values of x. Now the points at which $y = 0$, are only found on the X-axis. It means then that the solutions for this equation are on the X-axis. Therefore, the points $(2, 0)$ and $(-4, 0)$ are the solution points for the function. Hence, the x values are 2 and -4.



These two points give the solution of the quadratic equation. They are indicated as x- values as follows: $x = -4$ and $x = 2$.

On the Y-axis, the curve cuts the axis at one point given as $(0, -8)$. The y-intercept is -8.



In any quadratic function, the y-intercept of the given function is the value of c (the constant) of the function given.

The vertical line that makes the parabola to be symmetrical is the axis of symmetry. It is given as an x -value.

From the graph, the parabola is symmetrical about the value $x = -1$. Therefore, axis of symmetry is $x = -1$.

The vertex is the point of intersection of the axis of symmetry and the parabola. Here from the graph, the point of intersection is $(-1, -9)$. Therefore, the vertex is $(-1, -9)$. Here the point of intersection explained in the above statement is actually the highest or lowest point of the graph. In the curve shown above, the vertex is the lowest point.

There are times when you are required to determine the solutions, axis of symmetry, vertex and the y-intercept of the quadratic function without being given a graph. In this case, the following procedure would help:

Given a function $f(x) = ax^2 + bx + c$, we can write this function in the form $f(x) = a(x-h)^2 + k$ by completing the square. You dealt with the skill of completing the square in grade 10 when you learnt solving of quadratic equations using the completing of the square method. It is the same method being used here.

$$f(x) = ax^2 + bx + c$$

Factorise a , the coefficient of x^2 from the first two terms

$$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a^2}$$

Add the square of half the coefficient of x ; $\left(\frac{1}{2} \times \frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a^2}$$

When you factorise the expression:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}, \text{ you get this : } \left(x + \frac{b}{2a}\right)^2$$

In general, for a quadratic function of the form $f(x) = a(x-h)^2 + k$, the axis of symmetry is $x = h$, and the vertex is at (h, k) .

From this it shows that:

Axis of symmetry is given by $x = -\frac{b}{2a}$, where **a** and **b** are values in the function given.

Consider the following example. This example will help understand how to find the axis of symmetry and the vertex of the function. The example so far was more general in the sense that there was no numerical figure used. The next example uses numerical figures.

Example 2

Given $f(x) = 2t^2 - 5t - 3$, determine the axis of symmetry and the vertex of the function.

Solution

First we equate the function to y

$$2t^2 - 5t - 3 = y$$

Remember that $f(x)$ is the functional form of y .
Hence, $f(x) = y$

We factor out 2 from the first two terms as follows

$$2\left(t^2 - \frac{5}{2}t\right) - 3 = y$$

$$2\left(t^2 - \frac{5}{2}t + \frac{25}{16}\right) - 3 - \frac{25}{16} = y$$

Add the square of half the coefficient of x ; $\left(\frac{1}{2} \times \frac{5}{2}\right)^2 = \frac{25}{16}$

$$2\left(t - \frac{5}{4}\right)^2 - 3 - \frac{25}{16} = y$$

$$2\left(t - \frac{5}{4}\right)^2 - \frac{73}{16} = y,$$

The expression in brackets reduce it to simpler form as shown below: $t^2 - \frac{5}{2}t + \frac{25}{16} = \left(t - \frac{5}{4}\right)^2$

This now is the form $a(x - h)^2 + k = y$, where h is the axis of symmetry and k is the vertex.

Therefore, axis of symmetry for the quadratic function above is:

$$x = \frac{5}{4} \text{ and the vertex is } y = -\frac{73}{16}$$



This activity is similar to the example above. It is a follow up to the example to check how much knowledge you have acquired in answering such questions. It will help you work through what you have learnt so far

Activity 1

Given $f(x) = x^2 - 6x + 5$, determine the axis of symmetry and the vertex of the function.

Write your answer below



This activity has been on finding the axis of symmetry and the vertex. You should have noticed that the knowledge on completing the square on solving quadratic equations is applied here. However, you will learn this in the next unit although we have tried to tackle it here in this section above.

You should have realised that the coefficient of x^2 is 1. Therefore, the factorisation does not work.

We complete the square by using half of the coefficient of x and squaring it then adding it as shown below.

$$f(x) = (x^2 - 6x + 9) + 5 - 9$$

$$f(x) = (x - 3)^2 - 4$$

$\frac{-6}{2} = -3$. Then, $(-3)^2 = 9$
This is the 9 we have added and subtracted as shown.

Therefore, the axis of symmetry is: $x = 3$ and the vertex is: $y = -4$.

We have dealt with finding the axis (line) of symmetry of the given graph and the vertex of that same graph. We have seen how we can actually use two methods of coming up with our symmetry line and the vertex. In the first method, we did it graphically, meaning that we used the graph after having to draw it. In the second method, we simplify a given quadratic function in to the $y = (x + h)^2 + k$, where h is the axis of symmetry and k is the vertex.

Now you are going to learn how to find the gradient of a curve of the quadratic nature. In this method you will use the knowledge of gradient of a straight line. It is very important. If you have difficulties, please revise that section before you go on to study this section.

Finding the Gradient of the Quadratic Function

In finding the gradient of the parabola (quadratic curve), we draw a tangent line to the curve at a particular point. A tangent line is a straight line that meets a curve at only one point. Since it is a straight line, the gradient formula is used to calculate the gradient.

Let us consider the following example given below. This example will help you understand how to calculate the gradient of a curve. You have already learnt and for some of you it was revision when you were learning finding the gradient of a straight line. Although it may seem different in that it is a curve, the end result is that you are using the same method of finding gradient of a straight line.

Example 3

Using the graph of the function $f(x) = x^2 + 2x - 8$ shown below, draw a tangent to the curve at $(-2, -8)$ and find the gradient of the curve at this point.

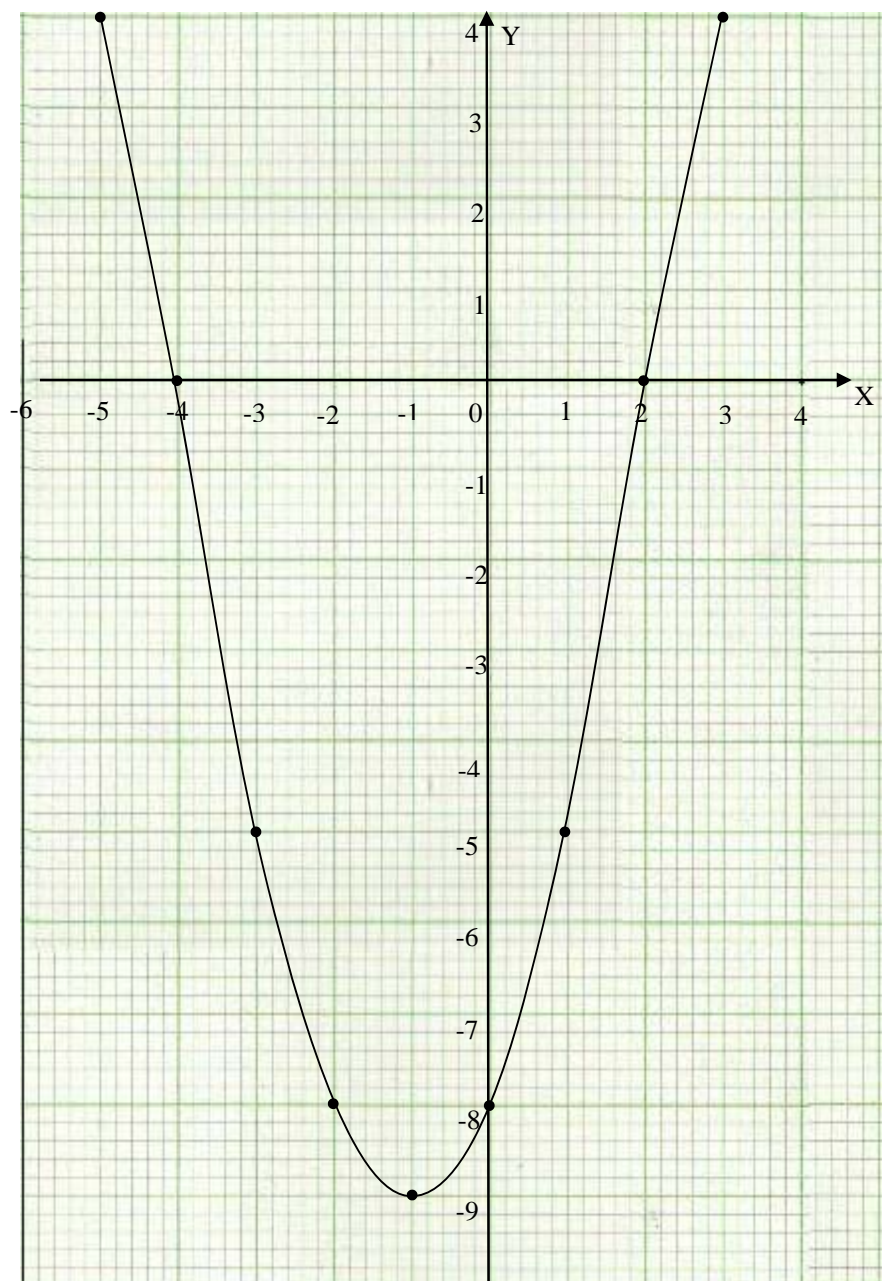


Figure 8: The graph of $y = x^2 + 2x - 8$

Solution

We are to find the gradient by first drawing a tangent line to the curve at $(-2, -8)$ on the curve.

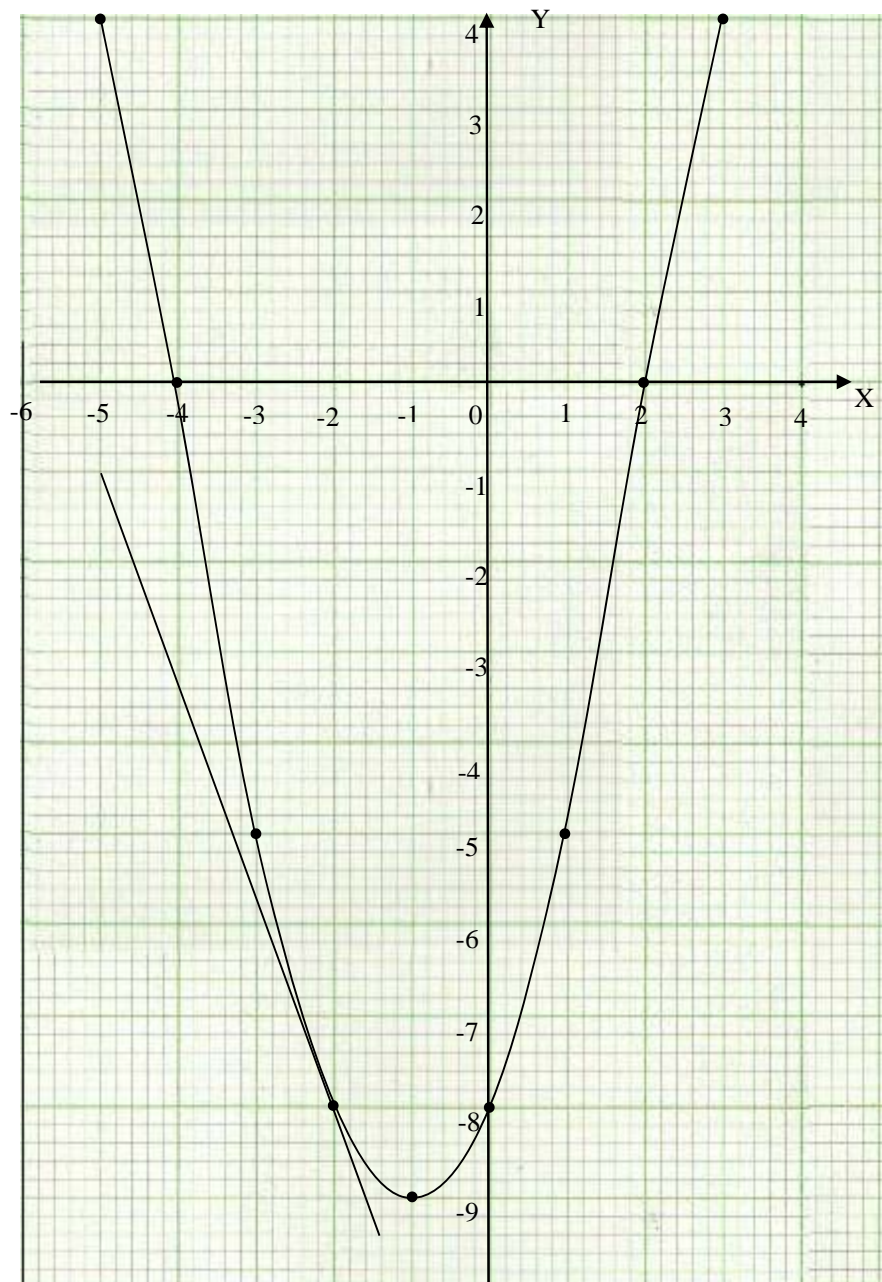


Figure 9: The graph of $y = x^2 + 2x - 8$

Having drawn the tangent line touching the curve at the point $(-2, -8)$, we now choose from the line two points to use in the calculation of the gradient.

From the graph, we pick $(-2, -8)$ and $(-5, -1)$;

Using the gradient formula of a straight line, $m = \frac{y - y_1}{x - x_1}$, we substitute the values from the points and calculate for the gradient.

$$m = \frac{y - y_1}{x - x_1}$$

$$m = \frac{-8 - (-1)}{-2 - (-5)} = \frac{-8 + 1}{-2 + 5} = \frac{-7}{3}$$

Hence the gradient of the line is $-\frac{7}{3}$.

Therefore, the gradient of the curve at $(-2, -8)$ is $-\frac{7}{3}$.



The gradient of the line is the same as the gradient of the curve at the point of intersection.



The next activity is to help you to calculate the gradient of a curve at a point given. It is very similar to the one already done above. The only difference is that it is cap shaped.

Activity 2

Using the given graph below, find the gradient of the curve at the point given as $(4, -4)$.

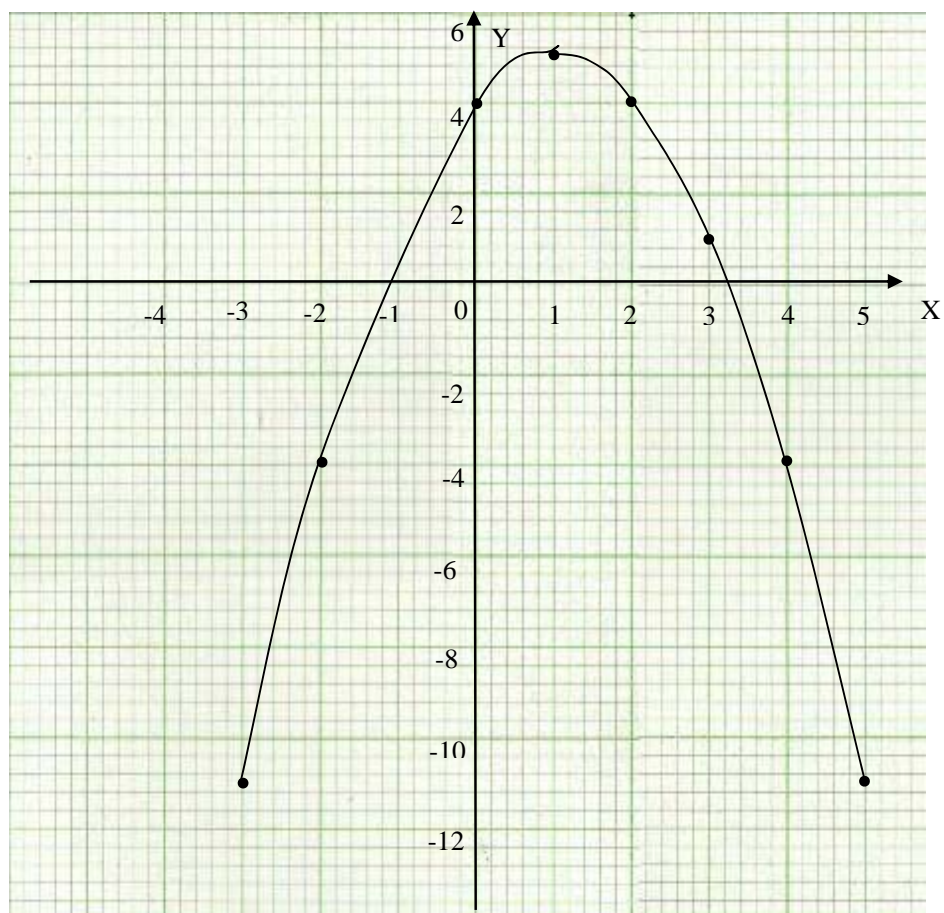


Figure 10: Graph of a quadratic function

Write your answer in the space provided below.



You were required to apply the knowledge of finding the gradient of a straight line using the formula. Go through the feedback after you have worked on your own and practice this method whenever you have a chance. Practice will help you master the method.

In answering such kind of questions, it is important for you first to identify the point of interest, in this case the point $(4, -4)$ at which the two intersect; that is, the curve and the tangent that you would draw on the graph.

Then draw the tangent as shown below.

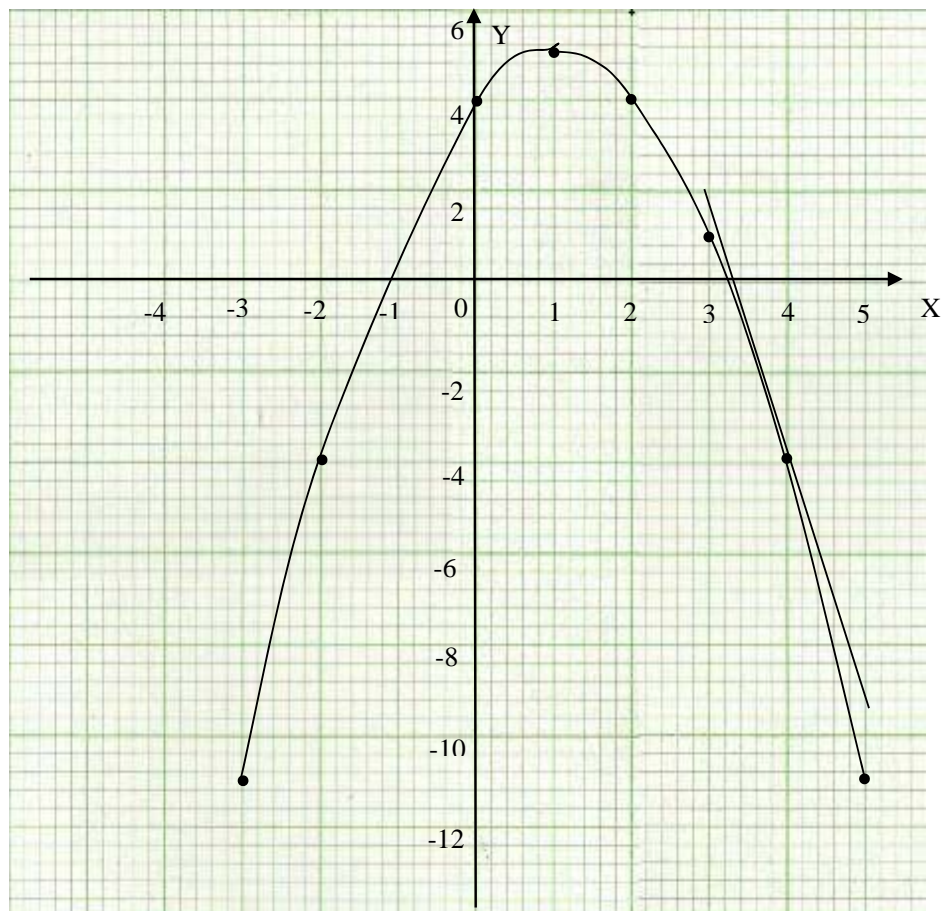


Figure 11: Solution - graph of a quadratic function with tangent drawn in

From the graph you pick out two points that would be used in calculation of the gradient. It is ideal and easy for you to pick points that consist of integer values. In this case we can pick the intersection point and another point.

We pick; (4, -4) and (3, 2), and then substitute them in the formula for calculating gradient as shown below:

$$m = \frac{y - y_1}{x - x_1} \text{ let } x_1 = 3, y_1 = 2, x = 4 \text{ and } y = -4.$$

$$m = \frac{-4 - 2}{4 - 3} = \frac{-6}{1} = -6$$

Therefore, the gradient of the curve at the point is -6

You have so far learnt how to find the axis of symmetry, vertex and the gradient of a given curve. You should now have realised that the axis of symmetry is a line that divides a shape into two halves that are exactly the same. Folding the shape along that line, the two halves fit exactly in the outline of each other. You should have also realised that the vertex is simply the turning point of a given curve. The last thing you learnt is that the gradient of a curve changes from point to point. Hence to calculate this gradient, you had to use the gradient of a line since it (gradient of a line) is the same at each point of the line.

You will now learn about cubic functions. These may be new to you but do not worry, you will enjoy the lesson. It will be explained to you and you will be expected to do some activities to understand the cubic functions.

Cubic Functions

As already illustrated, a cubic function is a function with the highest power of x being 3. Examples are of the nature $y = x^3 + x^2 + x + 2$. In this example, the highest power is a 3 and the constants (coefficients of the x variable) are all ones. Hence, such a function is a cubic function. A cubic function can be written in several forms. The key point to note is that the highest power should be 3 in the given function.

As for these functions, we will basically consider the simpler types and in this we will only do drawings of some simpler graphs of cubic functions and use the graphs to determine the minimum or maximum points of the curves.

The example below is for you to learn how to draw a cubic graph. It will introduce you to cubic graphs and their nature.

Example 4

Draw the graph of $y = x^3$.

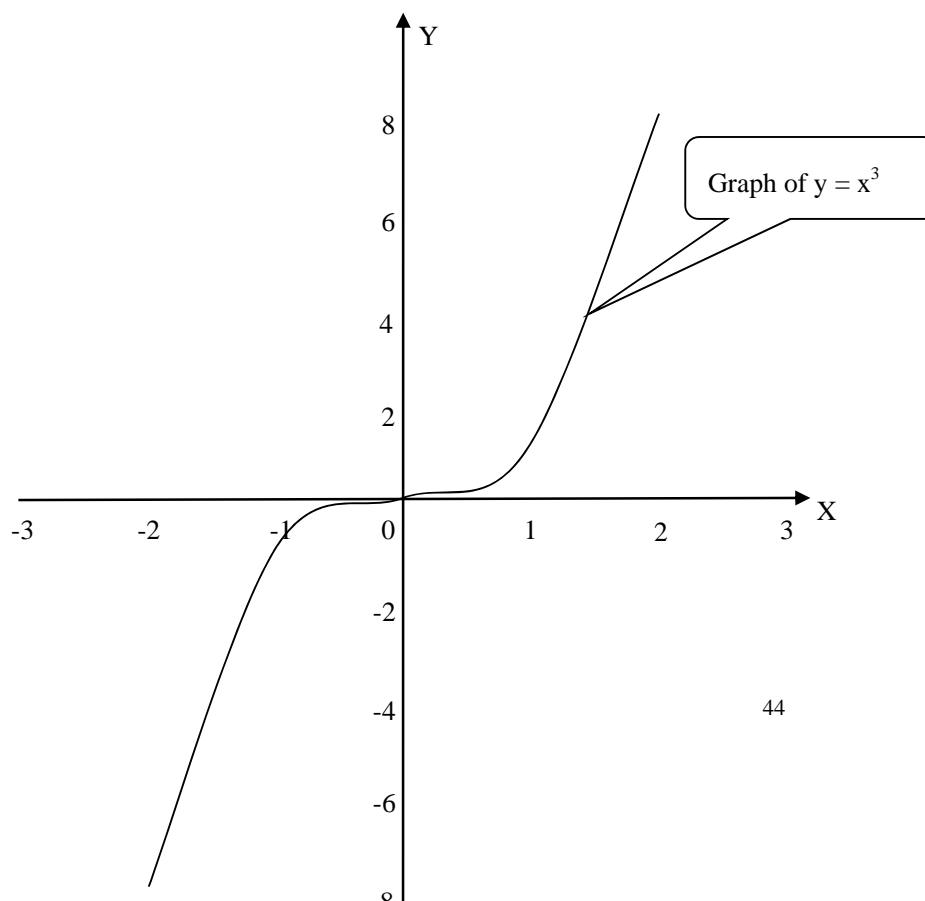
Solution

We first consider the table of values for the x values from -3 to 3.

X	-2	-1	0	1	2
y	-8	-1	0	1	8

Table 4: The values of the function $y = x^3$.

So this graph will appear as shown below:



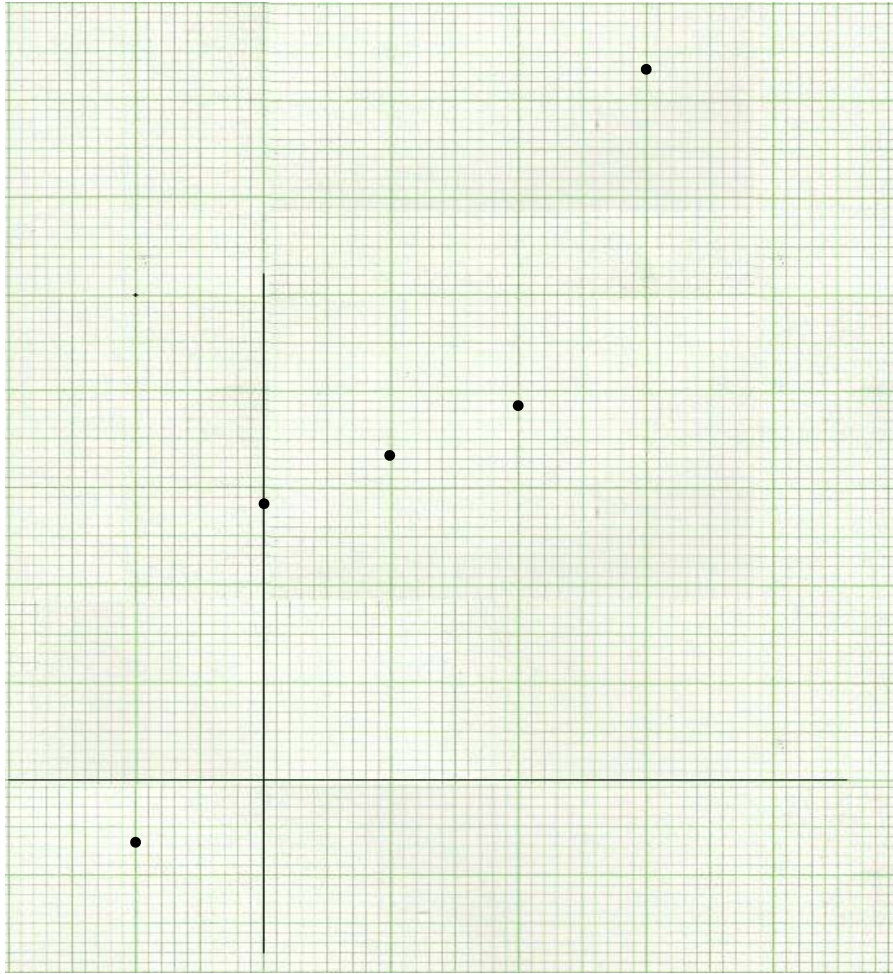


Figure 12: The graph of $y = x^3$

These functions are usually meant to introduce a student to cubic function. In the syllabus, you are expected to learn to draw these graphs and determine the minimum and maximum values of the curves.



The following activity is for you to check your knowledge on drawing the graphs and determining the nature of the turning points. As earlier stated, a curve of cubic function has two turning points of which one is a maximum and the other is a minimum. You are expected to determine which one is a maximum and which one is a minimum.

Activity 3

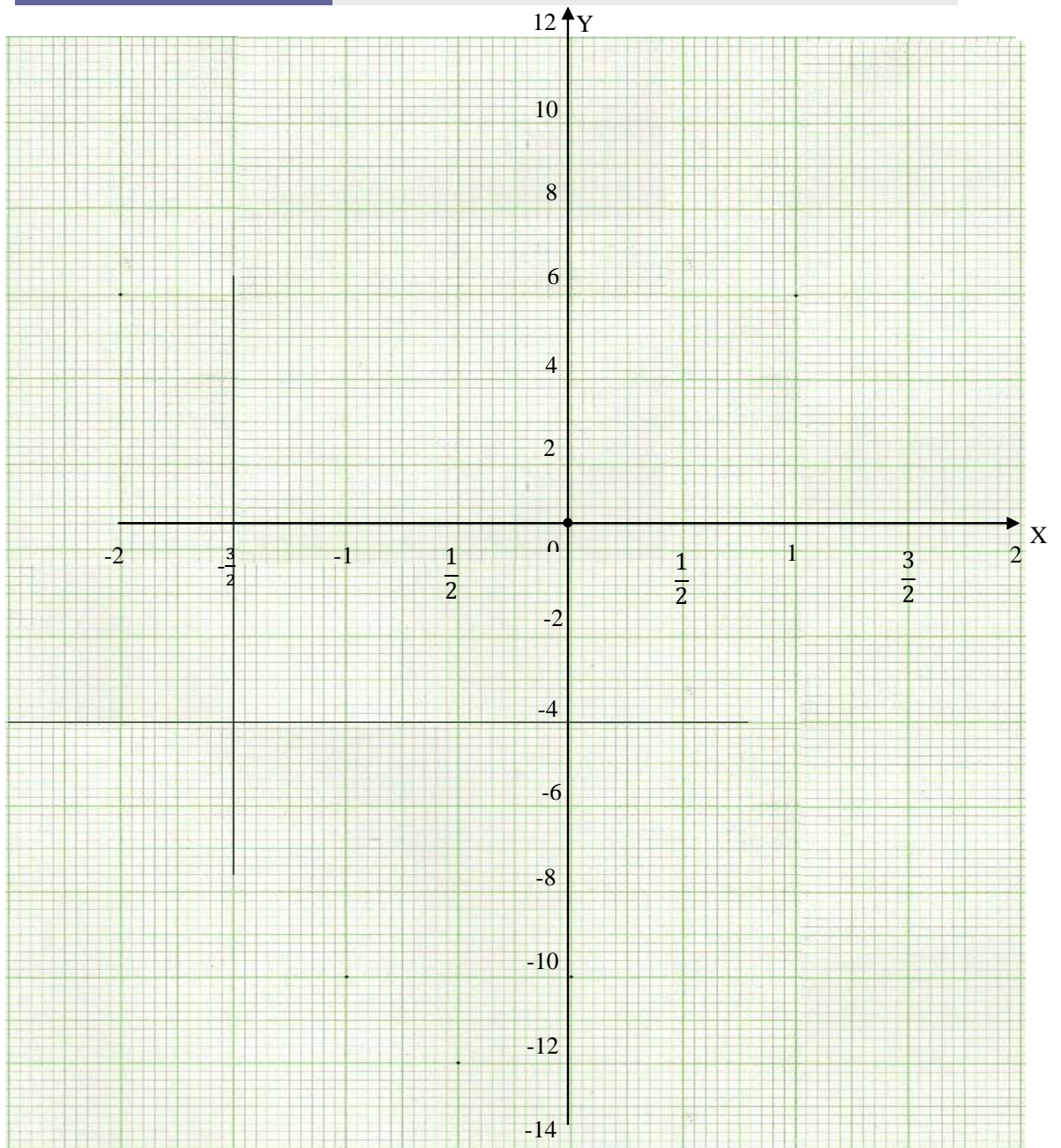
Draw the graph of $y = 2x^3$ for the x values from -2 to 2.

Complete the table

X	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y									

Table 5: uncompleted table for $y = 2x^3$.

Use the grid below to draw the graph.



The activity was for you to draw the graph. Steps necessary to draw the graph were done in the example. I am sure you applied them here.

The first thing we do is to find the values of y using the values of x given above. We use the table of values as shown below.

To see the proper behaviour of this curve, we include in the fractional values of x :

The values were calculated by substituting the values of x in the function and solving for the values of y . Let us use the fractional values to show the procedure of calculating the values of y :

$$\text{When } x = \frac{3}{2}, y = 2 \times \left(-\frac{3}{2}\right)^3 = 2 \times -\frac{27}{8} = -\frac{27}{4}$$

$$\text{When } x = \frac{1}{2}, y = 2 \times \left(-\frac{1}{2}\right)^3 = 2 \times -\frac{1}{8} = -\frac{1}{4}$$

You will notice that when we substitute the positive numbers, the answers are the same and positive as shown in the table.

X	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	-16	$-\frac{27}{4}$	-2	$-\frac{1}{4}$	0	$\frac{1}{4}$	2	$\frac{27}{4}$	16

Table 6: The completed table of $y = 2x^3$

After all your work and the points are plotted on the XOY-plane, the graph of $y = 2x^3$ should appear like the one below:

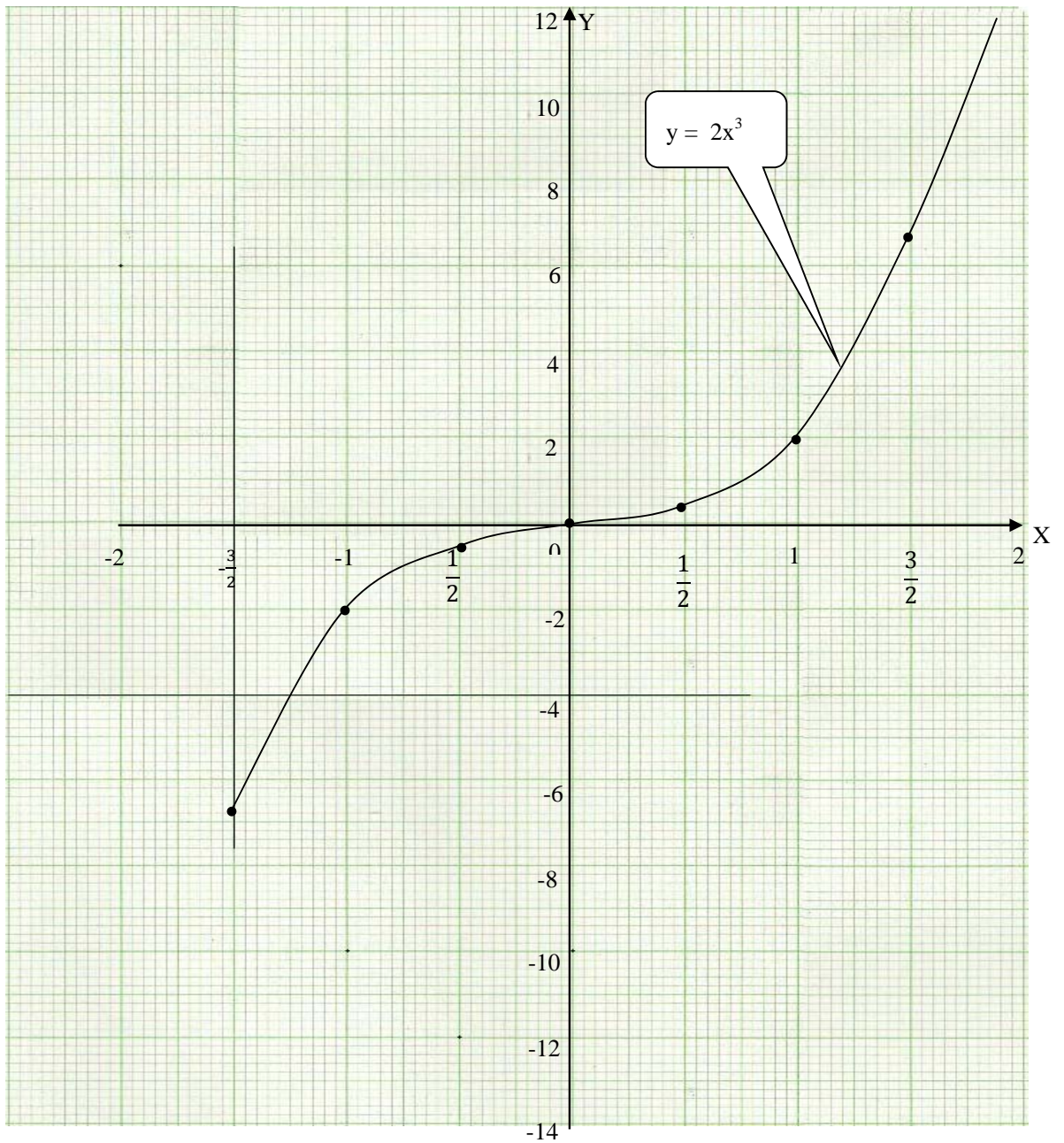


Figure 13: The graph of $y = 2x^3$

The above activity was for you to be able to learn and put into practice the knowledge and skill of drawing a cubic curve.

Now the next one is for you to master the skill of determining the minimum or maximum points of the curve after you have drawn it. You are expected to apply the skill of drawing already learnt and then practice determining the maximum or minimum points. The graph is similar to the above graph. However the graph will be more curved out than the above for the purpose of maximum and minimum value estimation.

Example 5

Draw the graph of $y = x(x - 2)(x + 2)$ for the x values from -3 to $+3$.

Using the graph, estimate

- The solutions of the graph
- The maximum and minimum values of the graph
- The value of x when y is maximum

Solution

As already stated in earlier examples, you first make the table of values for the domain given. The domain is a set of initial input values for the function. By now, you should know how to calculate these values. As a reminder, here is some work for just one value:

We will take the values of x as: $-3, -2, -1, 0, 1, 2,$ and 3

When we take the value of x as -3 , we substitute it in the function equation and then solve the value of y as follows:

$$X = -3,$$

$$Y = -3(-3 - 2)(-3 + 2)$$

$$Y = -3(-5)(-1)$$

$$Y = -3(5)$$

$$Y = -15$$

Usually students confuse:
 $-3-2$ as $-3 \times (-2)$ Please take note that: $-3 - 2$ is simply adding the two values:

Using the above shown procedure, we calculate all the remaining values of y using the other remaining values of x given in the domain.

Having calculated the values of y , the table will be as shown below:

X	-3	-2	-1	0	1	2	3
Y	-15	0	3	0	-3	0	15

Table 7: showing the values of x and y

When these values are plotted on the XOY plane and the graph is drawn, the graph is as shown below:

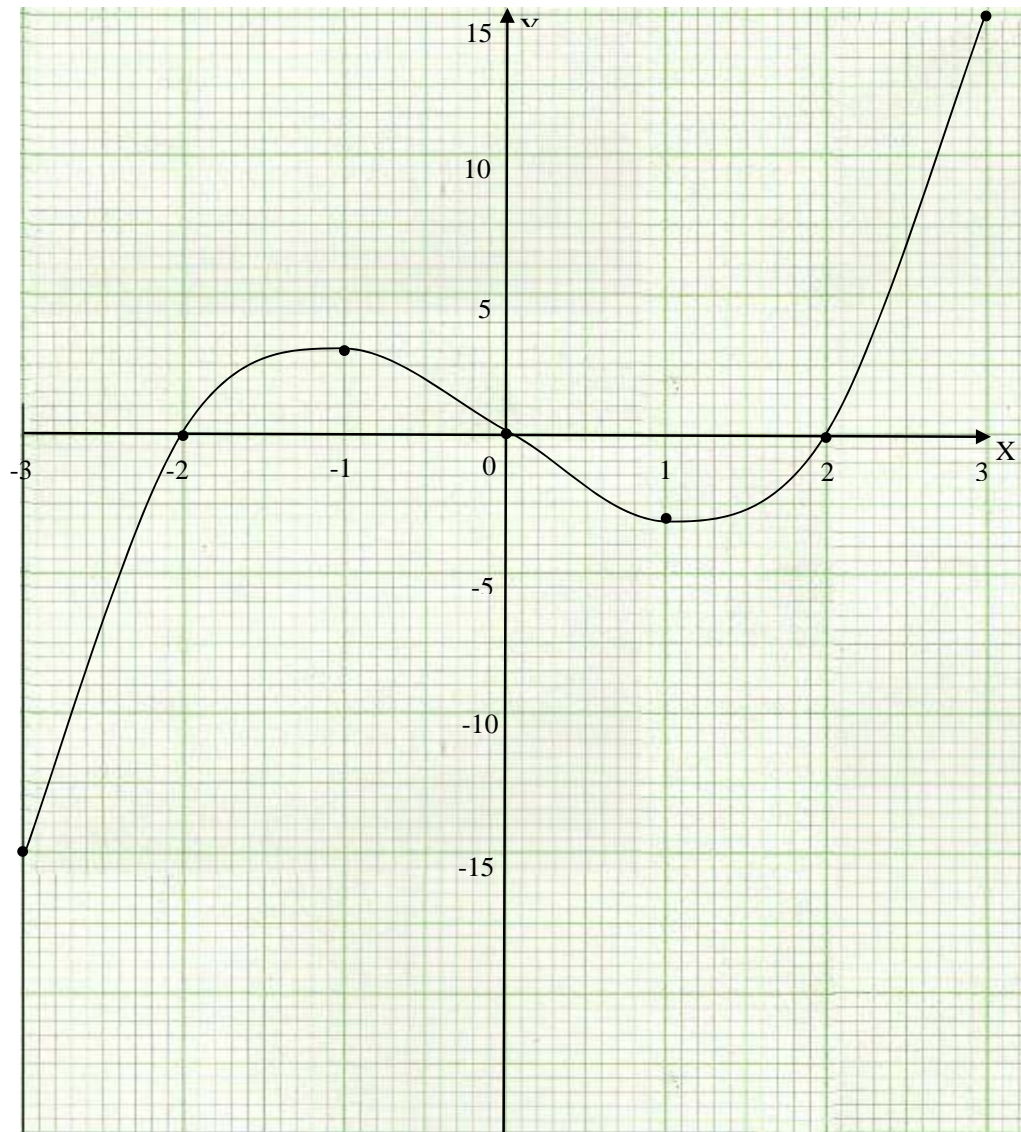


Figure 14: The graph of $y = x(x - 2)(x + 2)$

- (a) The solution of the graph is always given by the points at which the curve cuts the X-axis as earlier stated in example 1 under the quadratic and cubic graphs. When you study the next unit on solving quadratic equation this concept will be explained in detail. However here we will try to explain the principle.

On the X-axis, the value of y is always zero. This means that you will be solving for x as y is already zero. Let us take our function and explain further.

The function is: $y = x(x - 2)(x + 2)$. But y is already zero ($y = 0$) we substitute the value of zero on y in the function and we get the following statement: $0 = x(x - 2)(x + 2)$ which can also be rewritten as: $x(x - 2)(x + 2) = 0$. At this point it means that we are solving for the x value.

So graphically, the values of x at which the curve cuts the X -axis are the roots for the solution. Using this principle, we read out the values at which the curve cuts the graph on the X -axis and we get the following values of x : $x = -2$, $x = 0$ and $x = 2$. These are the solutions for the graph.

- (b) The maximum and the minimum values refer to the highest point value of y and the lowest point value of y on the graph. As earlier stated, the maximum or minimum point is a turning point of the graph. In this case, the curve has two turning points as: $(-1, 3)$ and $(1, -3)$. Please note that when we say maximum or minimum point, we refer to the point as (x, y) . However, when we say the maximum or minimum value, we are referring to the value of y in the point.

The point stated as $(-1, 3)$ is the maximum point. From this point, the value of y is 3. Therefore the maximum value is 3.

The point stated as $(1, -3)$ is the minimum point. From this point, the value of y is -3. Therefore, the minimum value is -3.

- (c) The value of x when y is the maximum is estimated by picking the maximum point and reading off the value of x . The maximum point is $(-1, 3)$. Therefore, the value of x is -1.



The following activity is to help you practice the skill of estimating finding the graph for the maximum or minimum values. This time the situation is from a direct life situation not just from the mathematics language. You are expected to apply your knowledge gained to solve this situation.

Activity 4

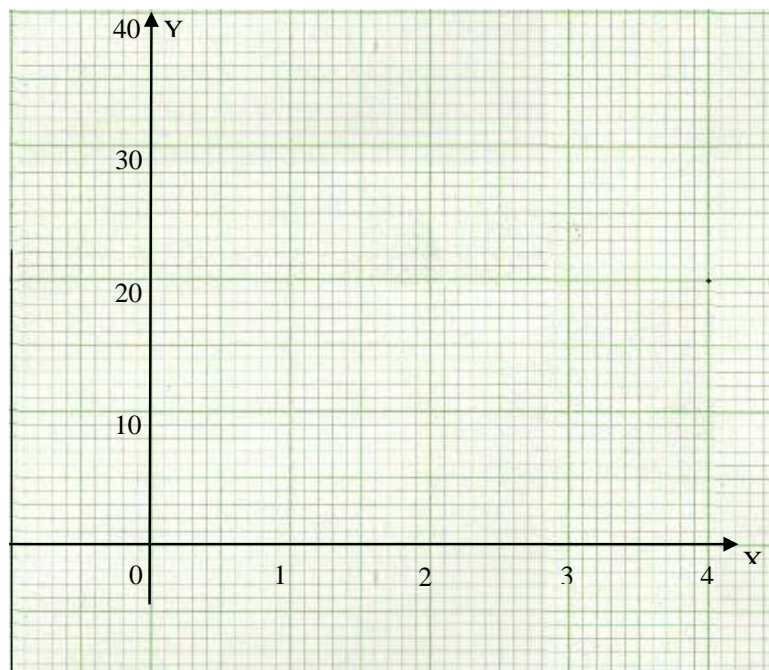
A skeleton box on a square base of side x cm is made from 36 cm of wire.

- (a) Find the height of the box in terms of x and hence show that its volume, V is given by $V = x^2(9 - 2x)$
- (b) Draw the graph of $V = x^2(9 - 2x)$ for the values of x from 0 to 4.

Using the graph, find the

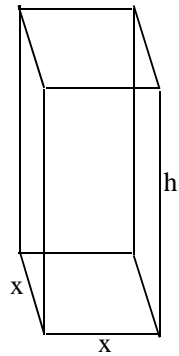
- (a) Maximum value of the box
- (b) The value of x when V is maximum

Use the space to show your calculations and the grid below to draw your graph.



In this activity knowledge of finding the area of a plane figure and the volume of a solid was very necessary. The basic knowledge you learn in grade 10 was sufficient. Check the feedback with your work after you have worked alone without assistance.

To answer this type of question, you need to sketch the frame of the box. Remember that it is done from wire.



- (a) *The way to answer this one is to consider perimeter. We know that perimeter is the distance around the figure. Here it is the sum of all the edges of this shape above.*

Perimeter = perimeter of base + perimeter of the top + perimeter of the sides

Perimeter of base = $x + x + x + x = 4x$.

Since the top is the same, perimeter of top is also $4x$

We then add the heights(sides) and get $4h$

Therefore, perimeter of shape = $4x + 4x + 4h = 8x + 4h$

We know the total length of wire is 36.

$$36 = 8x + 4h$$

We make h the subject of the formula

$$4h = 36 - 8x$$

$$h = 9 - 2x$$

Divide both sides by 4 and get this

Having found the height as $h = 9 - 2x$, we have to find the volume of the box. We are using volume because we have been asked to talk about volume.

Volume of the box = width \times length \times height

You should know by now that area of a rectangle or square is given as: $A = \text{width} \times \text{length}$. This means that volume = area of base \times height

Since the base has same length of sides, the area is: $A = x^2$
and we know that $h = 9 - 2x$.

Therefore, the volume = $A \times h$

Volume = $x^2 (9 - 2x)$. Hence we have shown.

- (b) To draw the graph, we first make the table of values by substituting each value of x given into the function and find the values of V as shown:

$$X = 0, V = 0$$

$$X = 1, V = 1(9 - 2) = 9 - 2 = 7$$

$$X = 2, V = 2^2(9 - 2 \times 2) = 4(9 - 4) = 4 \times 5 = 20$$

$$X = 3, V = 3^2(9 - 2 \times 3) = 9(9 - 6) = 9 \times 3 = 27$$

$$X = 4, V = 4^2(9 - 2 \times 4) = 16(9 - 8) = 16 \times 1 = 16$$

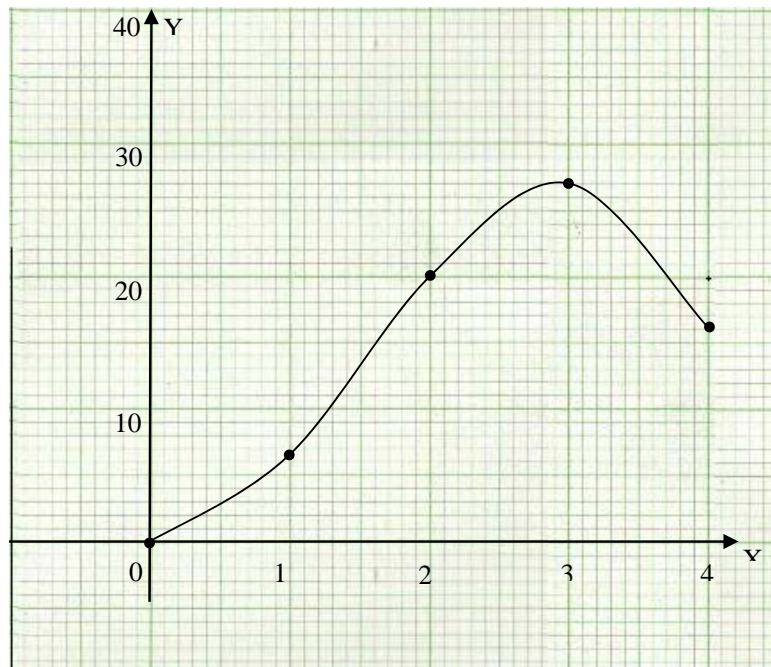
Remember to
multiply first before
subtracting

The table of values is now:

X	0	1	2	3	4
V	0	7	20	27	16

Table 8: Values of x and V

The graph will only have the positive part as illustrated from the table:



The graph is only on the positive side of the XOY plane because we do not have negative volume in real life. Volume would start from 0 and then increase but never negative.

(i) The maximum volume of the box is given when $x = 3$ as seen from the graph. Therefore, the maximum volume is 27cm^3

(ii) When V is maximum, the dimensions are: width = 3cm, length = 3cm and height = 3cm.

Topic Summary



You have learned in this topic how to find the x- and y- intercepts of a given quadratic graph. You also learned to find the gradient of a quadratic graph. You went on to learn how to draw the graph of a quadratic graph and the interpretation of the same. You also learnt how to draw the cubic function graphs and how to interpret the graph given. In both types of graphs, you also learnt how to find the minimum and maximum points.

We also noted that:

- The equation of the quadratic graph is written in intercept form of $y = a(x - p)(x - q)$, where p and q are the x-intercepts.
- The gradient of a quadratic graph is equal to the gradient of a tangent to the curve at the point of intersection.
- The gradient can be calculated using any two points on the tangent line.
- The equation of the cubic graph is written as $y = ax^3 + bx^2 + cx + d$, where a , b , c and d are constants.
- The gradient of the cubic is calculated in the same way as calculating the gradient of the quadratic function.

You were also encouraged to answer all the questions in the activities provided in this topic. The activities were intended to help you assess how well you understood the content of the various sub-sections of the topic. If you did not do very well in the activity, this means that you needed to go over the material again. Remember that the activities were meant for you to achieve the topic outcomes. You had to draw the quadratic and cubic function in the activities. You also had to find the gradient of the curve at a given point and interpret the graphs of the function. Your performance shows you how much knowledge and skills you have acquired.

In the next topic we will discuss the area under the curve.

In summary you have learnt how to draw the graphs of quadratic functions, finding the gradient and even interpreting the graphs. You have also learnt how to draw cubic functions. In order to know how much you have learnt in this topic, you can do the topic exercise 2 which appears at the end of the unit in the assignment section. Follow the same method of checking your answers as recommended in topic 1. If there are still some sections that you do not understand, discuss these with your tutor and other students.

Topic 3 Area under the Curve

This topic is the third and last topic of the unit. You have learnt how to draw the graphs of function in the previous topics. You were also taught how to interpret a given graph. You will now learn how to find the area under a given curve. You will learn how to use the trapezium to calculate the approximate value of the areas under the given curves. The method may be relatively new to you as you might not have learned this in the Junior Certificate Program. However when you dealt with finding areas of irregular shapes in your junior program and grade 10 work, you dealt with this method of finding areas. In your earlier work, you used squares to find the area of a given shape. In this grade 10 work, you used a grid and placed it on the figure of interest. You then counted the squares covering the figure. The total number of squares, including the half and almost whole squares, gave you the estimated area of the figure.

This time you will learn to draw the trapezium and use it to find the areas required. You should have dealt with the area of a trapezium when you dealt with areas of plane figures in grade 10 and in your junior program. Mind you, the area you will calculate will be approximated values of the actual area.

After studying the topic you will, once again, be required to answer self-assessment questions at the end of the topic. You are encouraged not to go through the feedback before doing the topic exercise.



We will address, in this topic, the last of the five unit outcomes, which is finding the estimated area under a curve.

To achieve this outcome by the end of the topic, you will be able to:



- Use the trapezium formula
- Calculate the area under a given curve.

Sometimes it is very difficult to determine the area under a given curve in which one cannot use a formula. In such situations, we use the approximation methods.

In the method we are to employ here, we consider the curve to be made up of trapezium shapes. By using these trapezium shapes, we are able to find an approximated area under the curve. As a reminder, a trapezium is a four sides figure which has one pair of sides parallel to each other as reflected in the diagrams of the examples. As you can see from the diagrams, the x value lines (such as $x = 2$ and $x = 3$) are parallel to each other.

Let us consider the following example. This example will help you understand the concept being used to calculate the area under the curve given. It will also help you understand a trapezium and what it looks like.

Example 5

Find the approximate value for the area under the curve $y = \frac{4}{x}$ between $x = 1$ and $x = 4$.

Solution

First in attempting this situation, we draw the graph and then draw lines parallel to the y -axis at $x = 1, 2, 3, 4$ to form the three trapeziums.



The area under the curve is approximately equal to the sum of the areas of the number of trapeziums drawn from the curve.

To avoid too much unnecessary work, we sketch the curve using the point values of x given.

The table of values is as shown below:

X	1	2	3	4
Y	4	2	$1\frac{1}{3}$	1

Table 9: values of $y = \frac{4}{x}$

The values of y in the table were calculated by substituting the value of x in the function. Let us do two substitutes to show the work.

$$\text{When } x = 1, y = \frac{4}{1} = 4$$

$$\text{When } x = 3, y = \frac{4}{3} = 1\frac{1}{3}$$

After all the points have been done and plotted on the graph, the curve is then drawn following the points as done in the other examples in this unit. So the graph will appear as shown below.

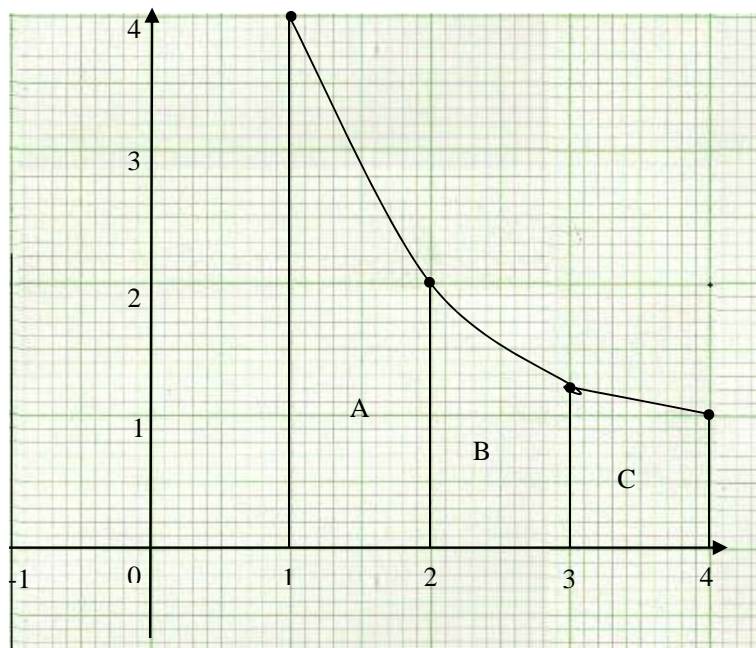


Figure 15: the graph of $y = \frac{4}{x}$

From the diagram, the trapeziums are A, B, and C. We are now to use the trapezium formula to find the areas of each trapezium.

$A = \frac{(a+b)h}{2}$, where **a** and **b** are the two opposite parallel sides and **h** is the perpendicular height between them.

For area A, $a = 4$ units, $b = 2$ units and $h = 1$ unit. So substituting these values in the formula:

$$A = \frac{(a+b)h}{2}$$

$$A = \frac{(4+2)1}{2} = \frac{6}{2} = 3 \text{ square units}$$

For area B, $a = 2$, $b = \frac{4}{3}$ and $h = 1$;

Remember the addition of fraction:

$$\left(2 + \frac{4}{3}\right) = \frac{2 \times 3 + 4}{3} = \frac{6+4}{3} = \frac{10}{3}$$

$$A = \frac{\left(2 + \frac{4}{3}\right)}{2} = \frac{\frac{10}{3}}{2} = \frac{10}{3 \times 2} = \frac{5}{3} \text{ square units}$$

For area C, $a = \frac{4}{3}$, $b = 1$ and $h = 1$;

$$A = \frac{\left(\frac{4}{3} + 1\right)}{2} = \frac{\frac{7}{3}}{2} = \frac{7}{3 \times 2} = \frac{7}{6} \text{ square units.}$$

We now combine the three areas by adding them to get the area under the curve between the values given.

$$\text{Area under curve} = 3 + \frac{5}{3} + \frac{7}{6} = \frac{35}{6} = 5\frac{5}{6} \text{ square units.}$$

This is the approximate area under the given curve.

From the above example, you should have noticed that it is not necessary to draw the full graph when dealing with the area under a curve. The curve part necessary for the problem solving is the one drawn.

The next example is similar to the one above. It is meant to emphasise the skills of solving such problems. The curve may appear different, but it is the same in nature and process.

Example 6

Find an approximate value for the area under the curve $y = \frac{12}{x}$ between $x = 2$ and $x = 5$.

- (a) State whether your approximate value is greater than or less than the actual value for the area.

Solution

As earlier stated, you need to draw the graph. From the information in the question, we need to draw the graph for values of x from 1 to 5. The values given from 2 to 5 are for the region to be considered under this discussion.

As usual we first make the table of the values of x and y by using the values of x from 1 to 5. Note that we have not used the value of x being 0. Would you know the reason? If so, write it down before proceeding.

To explain the reason, consider the following:

When $x = 0$, the division is not possible. What is zero divided into any number? What is: $y = \frac{12}{0}$? The answer is that it cannot be explained. Such a situation is called an undefined situation. Hence we will not know at which point the curve will cut the y axis. In this situation the curve moves along the axis but does not cut or meet the axis. When you have this kind of situation, you begin at some point somewhere. You will notice that in all the graphs of this nature, I mean division by zero, we are not starting at $x = 0$. Hence we start at $x = 1$

When $x = 1$, $y = \frac{12}{1} = 12$

When $x = 2$, $y = \frac{12}{2} = 6$

When $x = 3$, $y = \frac{12}{3} = 4$

When $x = 4$, $y = \frac{12}{4} = 3$

When $x = 5$, $y = \frac{12}{5} = 2\frac{2}{5}$

Division of 12 by 5 gives 2 and remainder 2 which is written as $2\frac{2}{5}$

From the above calculations, we make a table of values:

X	1	2	3	4	5
---	---	---	---	---	---

y	12	6	4	3	$2\frac{2}{5}$
---	----	---	---	---	----------------

Table 10: values of x and y

Using these values we now sketch the part of the graph that we need to use in answering our questions.

After drawing the graph, you need to draw the lines representing the area of interest. We are told that we need to find the approximate value of the area between $x = 2$ and $x = 5$. These lines are actually parallel to the y axis at $x = 2$, $x = 3$, $x = 4$ and $x = 5$. When these lines are drawn, the trapezium shapes are formed on the graph. Then we label the trapezium as A, B, and C. The graph will look like this:

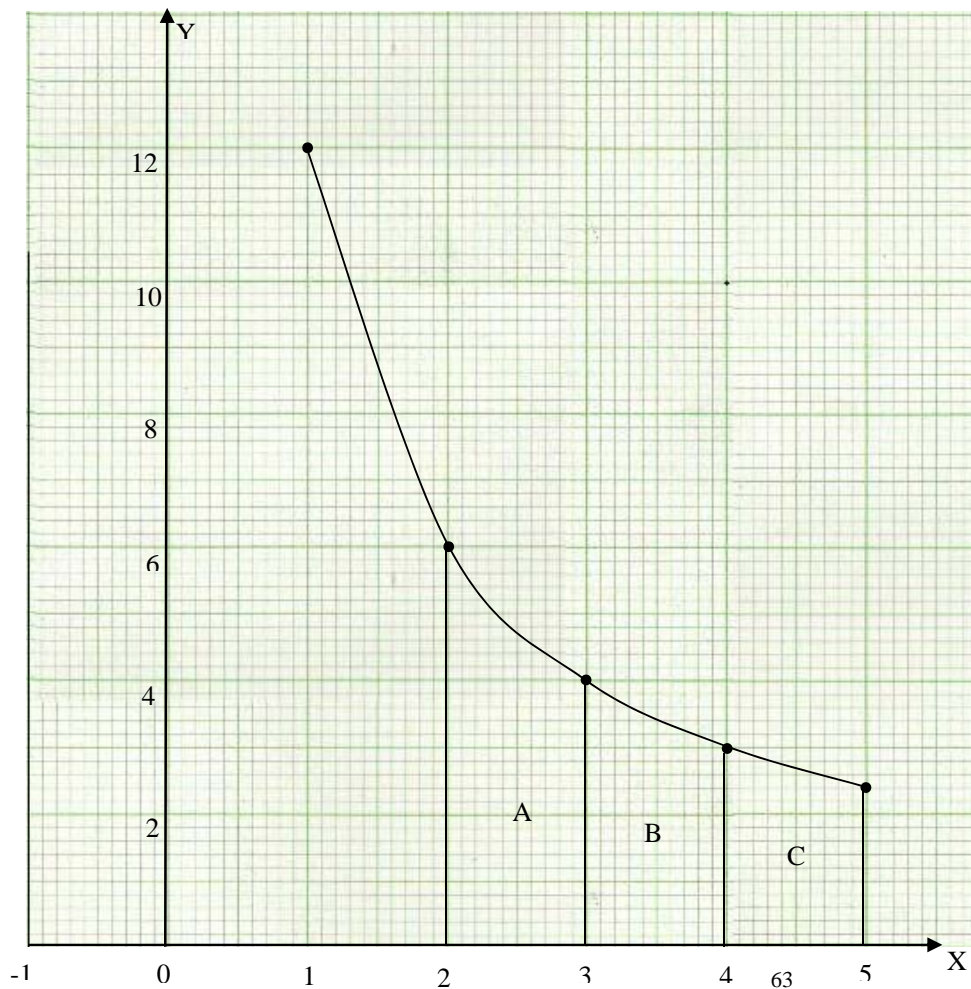


Figure 16: The graph of $y = \frac{12}{x}$

From the diagram, the figure making the area A or area B or area C is a trapezium. In a trapezium, the longer side is **a** and the shorter side is **b**. Take note that these are lines parallel to the y axis. The height of the trapezium is given by the distance between the two values on the x axis. The distance from 2 to 3 or 3 to 4 or 4 to 5 is the height, **h**.

With that information, you should note that the values of **a** and **b** are given by the values of y calculated already in this example. For each trapezium, we use the same values by identifying which is the longer side and which is the shorter side.

The formula for finding the area of a trapezium is given by the following: $A = \frac{1}{2}(a + b)h$

$$\text{For area A} = \frac{1}{2}(6 + 4) \times 1 = \frac{1}{2}(10) = 5 \text{ sq. units}$$

$$\text{For area B} = \frac{1}{2}(4 + 3) \times 1 = \frac{1}{2}(7) = 3\frac{1}{2} \text{ sq. units}$$

$$\text{For area C} = \frac{1}{2}\left(3 + 2\frac{2}{5}\right) \times 1 = \frac{1}{10}(27) = 2\frac{7}{10} \text{ sq. units}$$

Total area under the curve \approx area A + area B + area C

$$\text{Total area under the curve} \approx 5 + 3\frac{1}{2} + 2\frac{7}{10} = \frac{112}{10} = 11\frac{2}{10}$$

Therefore, total area $\approx 11\frac{1}{5} = 11.2$ square units.

- (a) The value of the area that we have calculated is less than the actual area of the area under the curve. When you look at the values used from the y-axis, they are in twos. That is the distance from one value to the other is 2 while on the x-axis the value used is 1 for the same distance. This means that on the x-axis we have undervalued the numbers. We should have used 2 also for each distance.

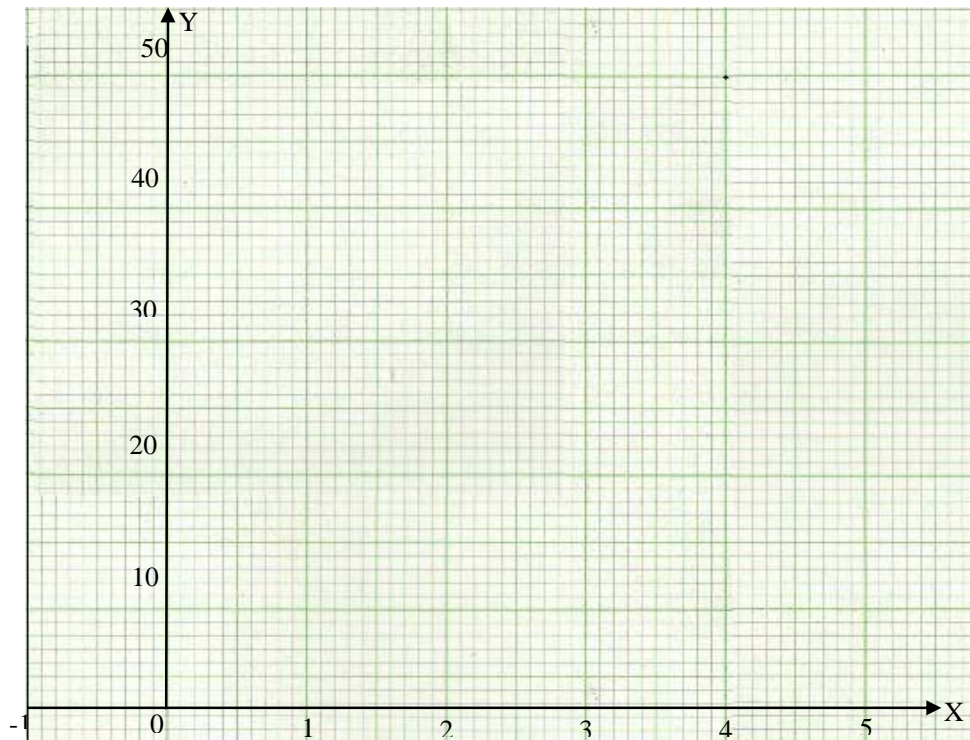


Having done the two examples here is an activity for you to practice what you have learnt so far. The activity is similar to the ones above already done. Take note that only the part of interest should be drawn to help solve the problem. You need to apply the knowledge you have learnt so far in this one.

Activity 5

- (a) Find an approximate value for the area under the curve $y = x^2 + 2x + 5$ between $x = 0$ and $x = 4$.
- (b) State whether your approximate value is greater than or less than the actual value for the area.

Use the grid below to draw your graph. Take the scale of 2 cm to 10 units on the y-axis and 2 cm to 1 unit on the x axis.



Grid 2



The activity was on finding the area under a curve. This required you to use the trapezium figure to estimate the value of the area. Here again it is a reminder that you dealt with a trapezium when you learnt about shapes in grade 10.

As earlier stated, you need to draw the graph first. The graph should have helped you to answer this question. From the information in the question, we need to draw the graph for values of x from 0 to 5. The values given from 0 to 4 are for the region to be considered under this discussion.

As usual we first make the table of the values of x and y by using the values of x from 0 to 5. These values of x that is; (0, 1, 2, 3, 4, and 5) are to be substituted in the function, $y = x^2 + 2x + 5$ to find the values of y as shown below:

$$\text{When } x = 0, y = 0^2 + 2(0) + 5 = 5$$

$$\text{When } x = 1, y = 1^2 + 2(1) + 5 = 8$$

$$\text{When } x = 2, y = 2^2 + 2(2) + 5 = 13$$

$$\text{When } x = 3, y = 3^2 + 2(3) + 5 = 20$$

$$\text{When } x = 4, y = 4^2 + 2(4) + 5 = 29$$

$$\text{When } x = 5, y = 5^2 + 2(5) + 5 = 40$$

From the above calculations, we make a table of values:

X	0	1	2	3	4	5
y	5	8	13	20	29	40

Table 11: values of x and y

Using these values we now sketch the part of the graph that we need to use in answering our questions.

After drawing the graph, you need to draw the lines representing the area of interest. We are told that we need to find the approximate value of the area between $x = 0$ and $x = 4$. These lines are actually parallel to the y axis at $x = 0$, $x = 1$, $x = 2$, $x = 3$ and $x = 4$. When these lines are drawn, the trapeziums shapes are formed on the graph. Then we label the trapezium as A, B, C and D. The graph will look like this:

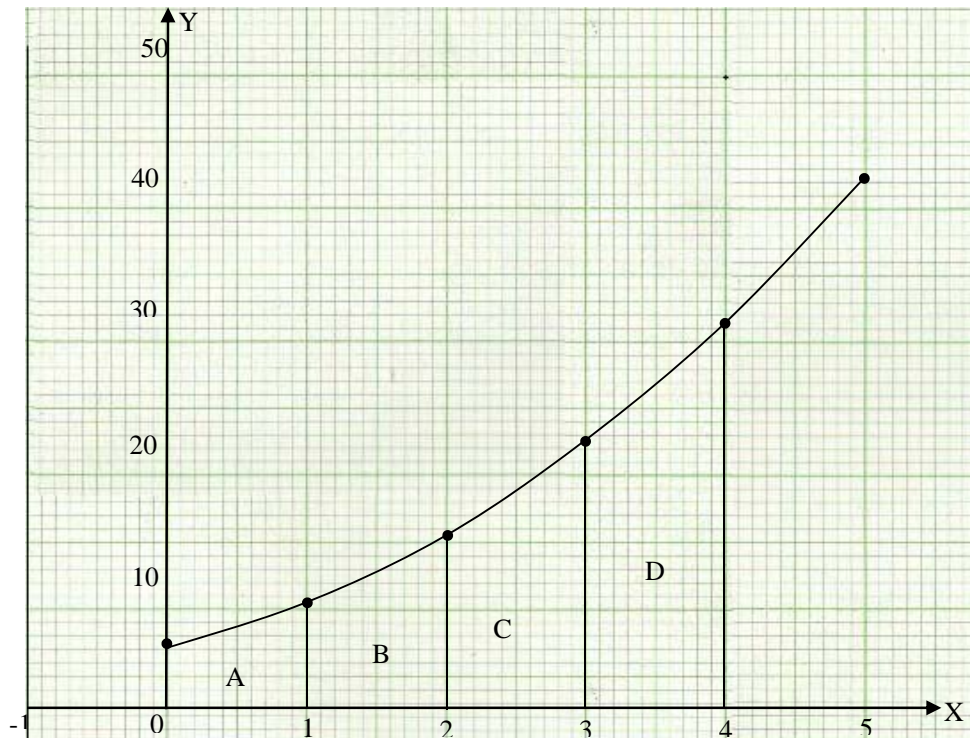


Figure 16: The graph of $y = x^2 + 2x + 5$

Again as in the example above, from the diagram, the figure making the area A or area B or area C or area D is a trapezium. The height of the trapezium is given by the distance between the two values on the x axis. The distance from 0 to 1 or 1 to 2 or 2 to 3 or 3 to 4 is the height, h .

With that information, you should note that the values of a and b are given by the values of y calculated already in this activity. For each trapezium, you should have used the same values by identifying which is the longer side and which is the shorter side.

The formula for finding the area of a trapezium is given by the following: $A = \frac{1}{2}(a + b)h$

$$\text{For area A} = \frac{1}{2}(5 + 8) \times 1 = \frac{1}{2}(13) = 6.5 \text{ sq. units}$$

$$\text{For area B} = \frac{1}{2}(8 + 13) \times 1 = \frac{1}{2}(21) = 10.5 \text{ sq. units}$$

$$\text{For area C} = \frac{1}{2}(13 + 20) \times 1 = \frac{1}{2}(33) = 16.5 \text{ sq. Units}$$

$$\text{For area D} = \frac{1}{2}(20 + 29) \times 1 = \frac{1}{2}(49) = 24.5 \text{ sq. Units}$$

Total area under the curve \approx area A + area B + area C + area D

Total area under the curve $\approx 6.5 + 10.5 + 16.5 + 24.5 = 48$ square units

Therefore, total area ≈ 48 square units.

- (a) *The value of the area that we have calculated is less than the actual area of the area under the curve. When you look at the values used from the axes, they are in twos. That is the distance from one value to the other is 10 while on the x-axis the value used is 1 for the same distance. This means that on the x-axis we have undervalued the numbers. We should have used 10 also for each distance.*

Topic Summary



In this third topic you have learned how to find the area under a given curve. You went on to learn how to use the trapezium drawn to calculate the approximate area.

We also noted that:

- It is difficult to find the area under the curve without an appropriate formula.
- One needs to draw the trapezium given the boundaries.

As in the first two topics, you were also encouraged to answer all the questions in the activity provided in this topic. The activity was intended to help you assess how well you understood the content of this topic. If you did not do very well in the activity, this means that you needed to go over the material again.

Now you have learnt how to estimate the area under the curve for given functions. You have also learnt how to use the trapezium shapes made for the region of interest. In order to know how much you have learnt you should do topic exercise 3 in the assignment section.

Unit Summary



In this unit there were three topics. In Topic 1 you learned to draw a straight line. This included identifying the x- intercept and the y- intercept of the line. You further learned how to find the gradient of the line and writing down the equation of the straight line.

In topic 2, you learned how to draw the quadratic and cubic function graphs. You went on to learn how to find the gradient of a curve at a point. You further learned how to interpret the graphs of quadratic and cubic function graphs.

In the third topic, you learned how to estimate the area under a given curve. You were required to use the trapezium figure or shape to estimate the area under a curve.

By now you would have also completed the topic 3 exercise. This means that, besides the activities that you did within the topics, you have assessed your progress at the end of each of the three topics.

Congratulations on completing the third unit of the mathematics 11 course. There is a tutor-marked assignment in this unit. We trust that the Tutor-marked assignment in addition to the activities and topic exercises will adequately help you to assess your own progress. You should complete and submit the tutor-marked assignment now. You will find it in the assessment section of this unit. Remember that this assignment is based on all the work you have done in units 1, 2 and 3. Feel free to revise these units before writing the assignment.

The next unit, unit 4 is on quadratic equations. You will learn how to solve the quadratic equations algebraically and graphically.

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Assignment



This consists of topic exercises from the three topics learnt in this unit. You expected to attempt all questions in all the exercises. All the areas covered in this unit have been covered by at least one question in this assignment.

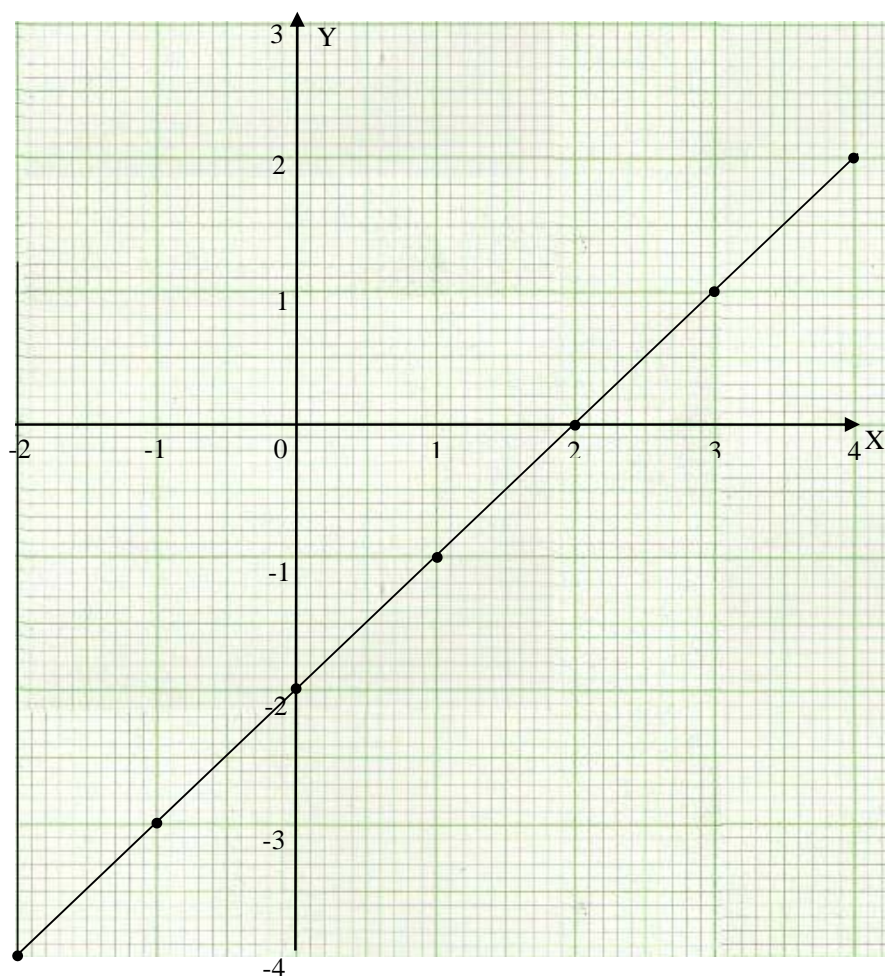
Exercise 1

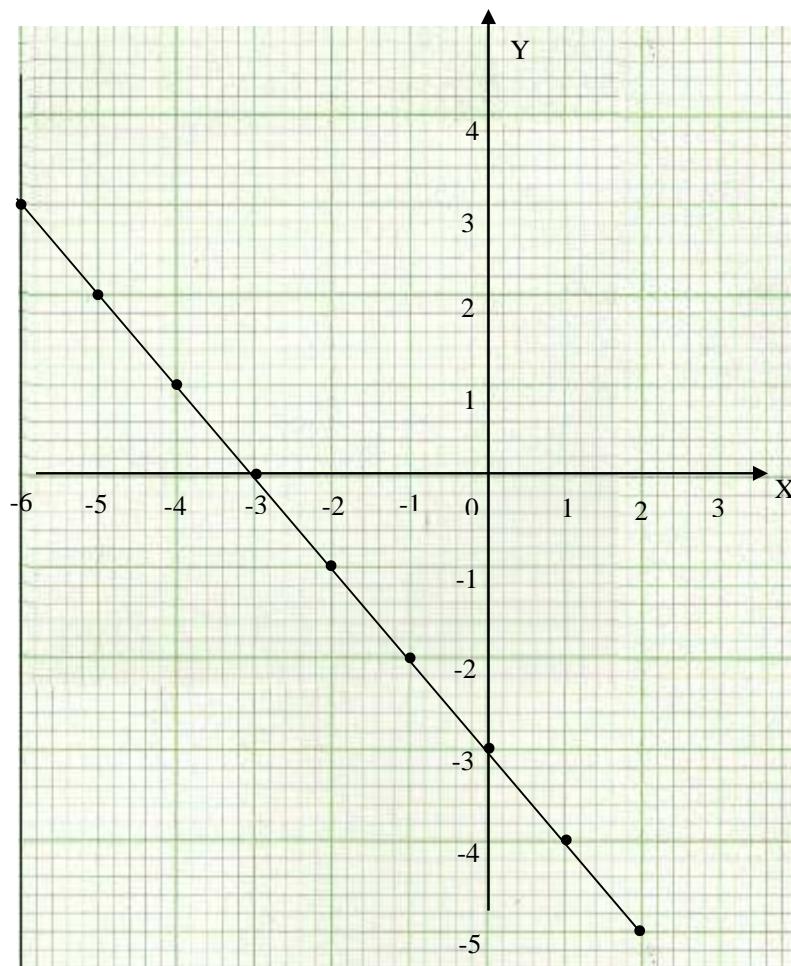
1. State the gradient (m) and the y-intercept of the following functions given below.
 - (a) $y = 3x - 6$
 - (b) $y = 12 - 3x$
 - (c) $y = \frac{1}{2}x + 4$
 - (d) $y = 8 - x$

2. Draw the graphs of the following linear functions on graph papers. Use a scale of 2 cm to represent 1 unit on both axes
 - (a) $y = 2x - 1$ for $-2 \leq x \leq 3$
 - (b) $y = x + 3$ for $-3 \leq x \leq 4$

3. Ms Mulenga's Car uses 1 litre of petrol for every 5 km travelled. She decides to travel a distance of 360 km.
 - (a) Draw the graph of Ms Mulenga's car fuel consumption.
 - (b) Use your graph to estimate the car consumption after 125 km.
 - (c) If she spent K38 270 on a 25 km journey, how far would she have to travel on fuel worthy K 187 500?

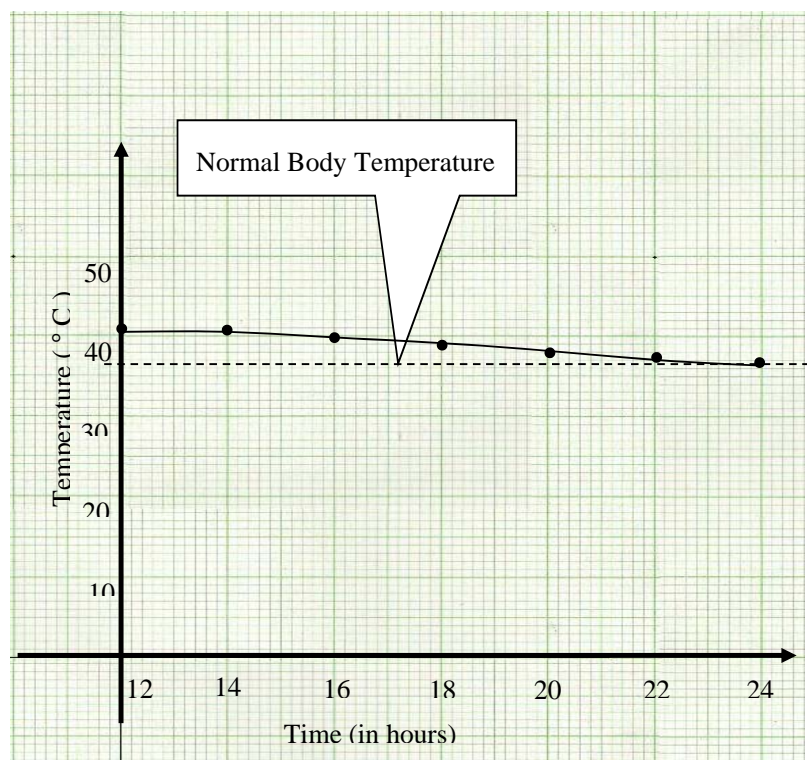
4. Find for the following graphs shown below;
 - (a) The gradient
 - (b) The equation of the function in the standard form; $y = mx + c$.





5. Matata was admitted to a hospital with high fever. She was admitted at 12 00 hrs with a temperature of 41°C .

Her temperature was monitored by nurses every 2 hours and recorded as shown in the graph below.



- (a) What was Matata's temperature after the first 2 hours?
- (b) Was there any improvement in Matata's illness? If so, explain.

Exercise 2

1. Draw the graph of the function $f(x) = -x^2 + 1$ for the domain $-2 \leq x \leq 2$, $x \in \mathbb{R}$.
2. Draw the graph of the function $y = 2 + 3x - x^2$ for the domain $-3 \leq x \leq 4$, $x \in \mathbb{R}$.
Use your graph to estimate:
 - (a) The value of y when $x = 0.5$
 - (b) The value of x for which $y = -1.5$
3. The height h metres of a stone thrown up from the ground after t seconds is given by the equation $h = -5t^2 + 25t$. Using a scale of 1 cm to represent 1 unit on the horizontal axis and 1 cm to represent 5 units on the vertical axis, draw the graph of the equation for $0 \leq t \leq 5$, $t \in \mathbb{R}$.
From your graph estimate;
 - (a) The time when the height of the stone above the ground is 10 metres
 - (b) The height of the stone above the ground at time $t = 4.5$ seconds
 - (c) The time when the stone hits the ground
4. Draw the graph of $y = x(x - 3)^2$ for values of x from -2 to 6 . Use a scale of 2 cm to represent 1 unit on the x -axis and 2 cm to represent 10 units on the y -axis and deduce from the graph,
 - (a) The values of x such that $x(x - 3)^2 = 0$
 - (b) The coordinates of the maximum turning point
 - (c) The coordinates of the minimum turning point
 - (d) The range of the function y

Exercise 3

1. Find an approximate value for the area under the curve $y = \frac{8}{x}$ between $x = 2$ and $x = 5$
2. Find an approximate value for the area under the curve $y = \frac{12}{x+2}$ between $x = 0$ and $x = 6$.

Answers to Exercises in the Assignment



This consists of topic exercises from the three topics learnt in this unit. You expected to attempt all questions in all the exercises. All the areas covered in this unit have been covered by at least one question in this assignment.

Exercise 1

1. Gradient (m) and y-intercept for the functions given
 - (a) $y = 3x - 6$
 $m = 3$; y-intercept is 6
 - (b) $y = 12 - 3x$
 $m = -3$; y-intercept is 12
 - (c) $y = \frac{1}{2}x + 4$

$$m = \frac{1}{2}; \text{ y-intercept is } 4$$

$$(d) y = 8 - x$$

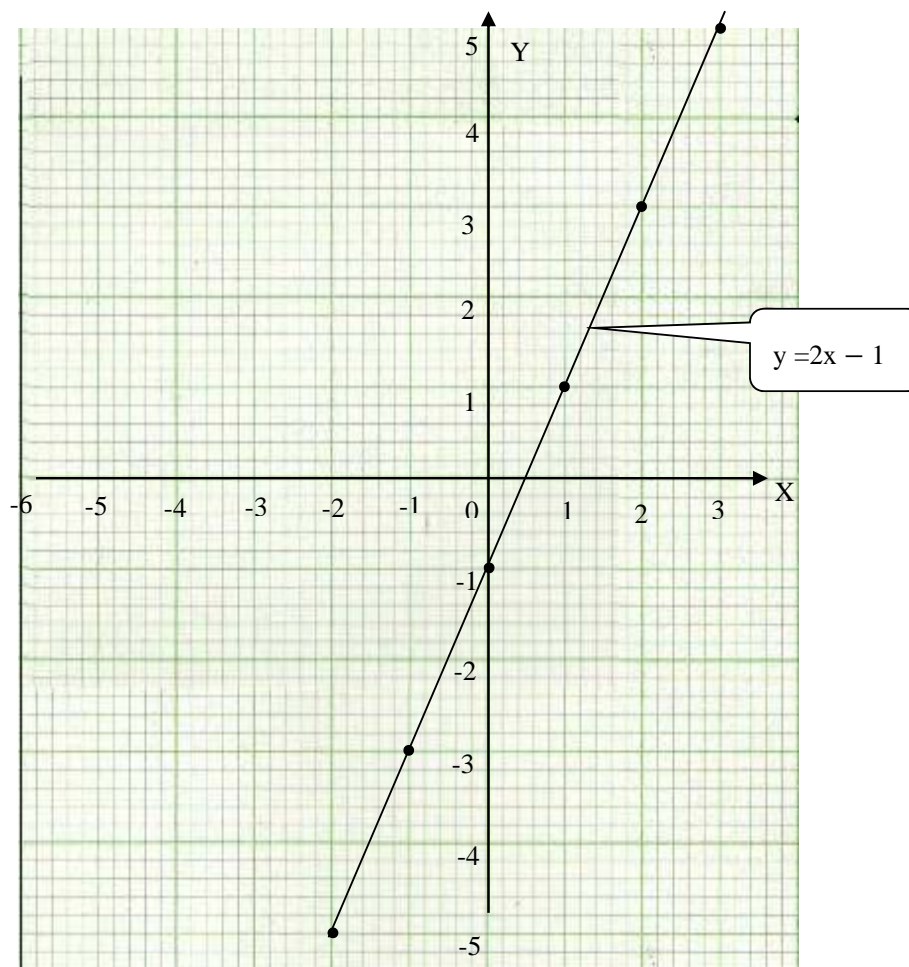
$$m = -1; \text{ y-intercept is } 8$$

2. The graphs are shown below.

$$(a) y = 2x - 1 \text{ for } x \text{ values from } -2 \text{ to } 3$$

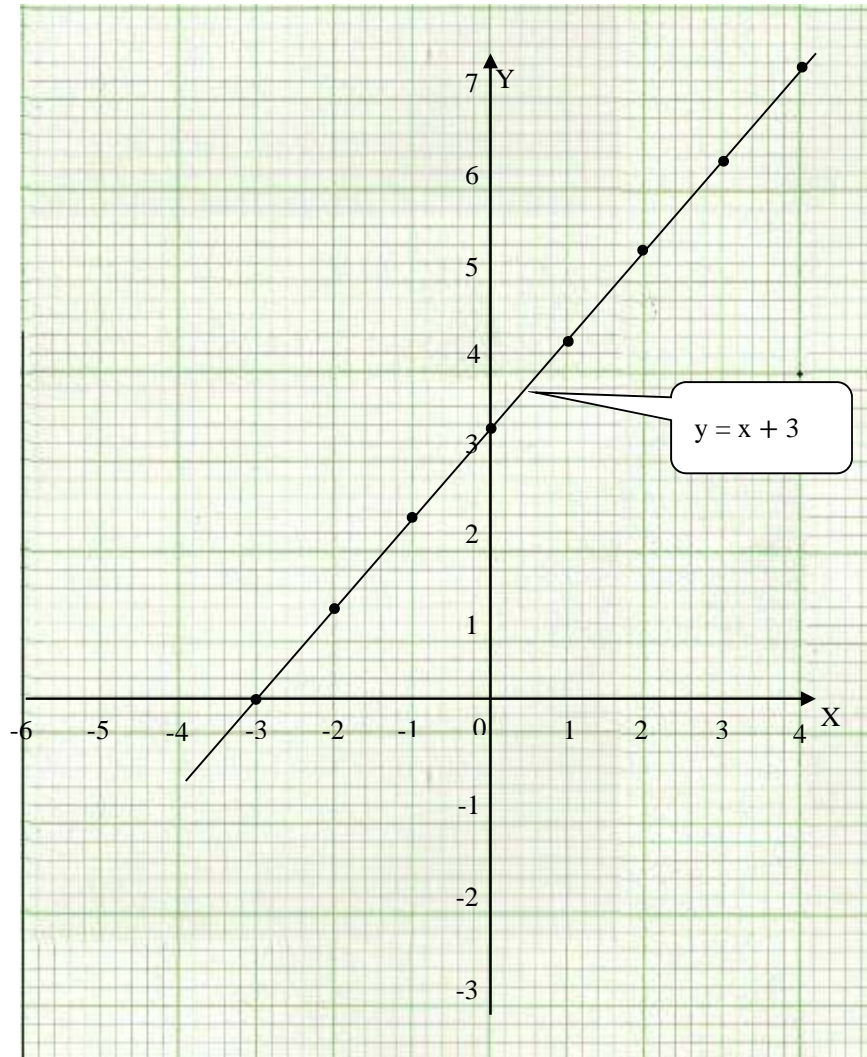
Table of values

X	-2	-1	0	1	2	3
y	-5	-3	-1	1	3	5

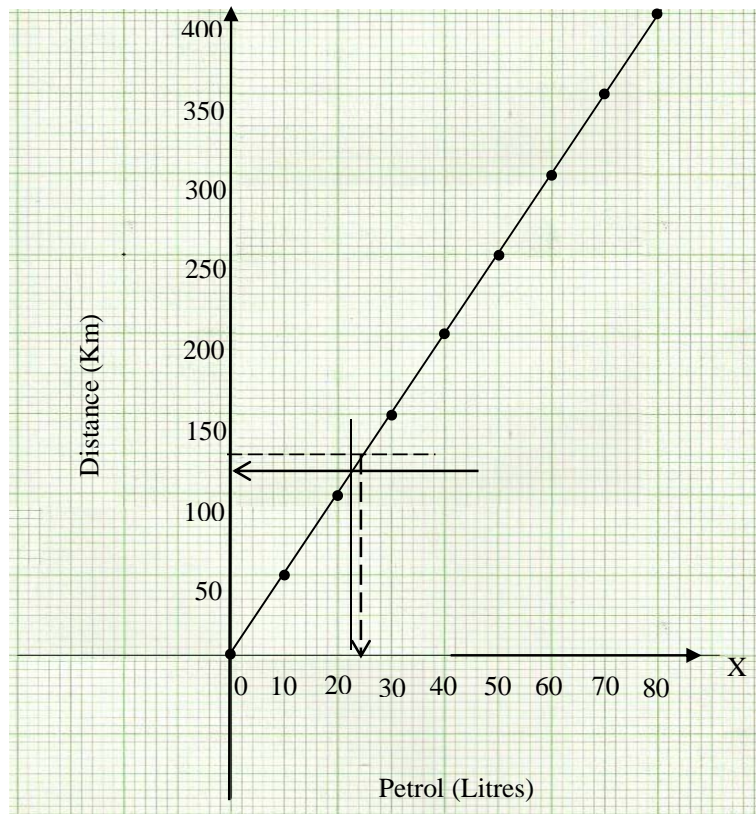


(b) $y = x + 3$ for x values from -3 to 4
Table of values

X	-3	-2	-1	0	1	2	3	4
y	0	1	2	3	4	5	6	7



3. The graph showing the fuel-Distance graph.



- (a) The graph is drawn as above
- (b) When the car has travelled 125km, it consumes approximately 25 litres of petrol
- (c) When we divide 38270 by 5 km, we get 5 litres. Meaning that 187500 divide by K7654 per litre gives 24.495 litres. Therefore, the car can cover approximately 122 Km.

4. The solution for the first graph is shown below:

(a) The gradient:

First we pick two points from the graph: (3, 1) and (1, -1)

$$M = \frac{y - y_1}{x - x_1}$$

$$M = \frac{1 - (-1)}{3 - 1}$$

$$M = \frac{2}{2} = 1.$$

The gradient is 1.

(b) The equation in the form $y = mx + c$

Since we know the gradient, $m = 1$ and the y intercept which is -2.

The equation is $y = x - 2$.

The solution for second graph is shown below:

(a) The gradient:

First we pick two points from the graph: $(-4, 1)$ and $(-2, -1)$

$$M = \frac{y - y_1}{x - x_1}$$

$$M = \frac{-2 - 1}{-1 - (-4)}$$

$$M = \frac{-3}{3} = -1.$$

The gradient is -1.

(b) The equation in the form $y = mx + c$

Since we know the gradient, $m = -1$ and the y intercept which is -3.

The equation is $y = -x - 3$.

5. (a) Matata's temperature after the first 2 hours was 39°C

(b) From the graph, it illustrates that there was a decline in the temperature as the line was gently sloping downwards.

This means that there was an improvement in her high temperatures.

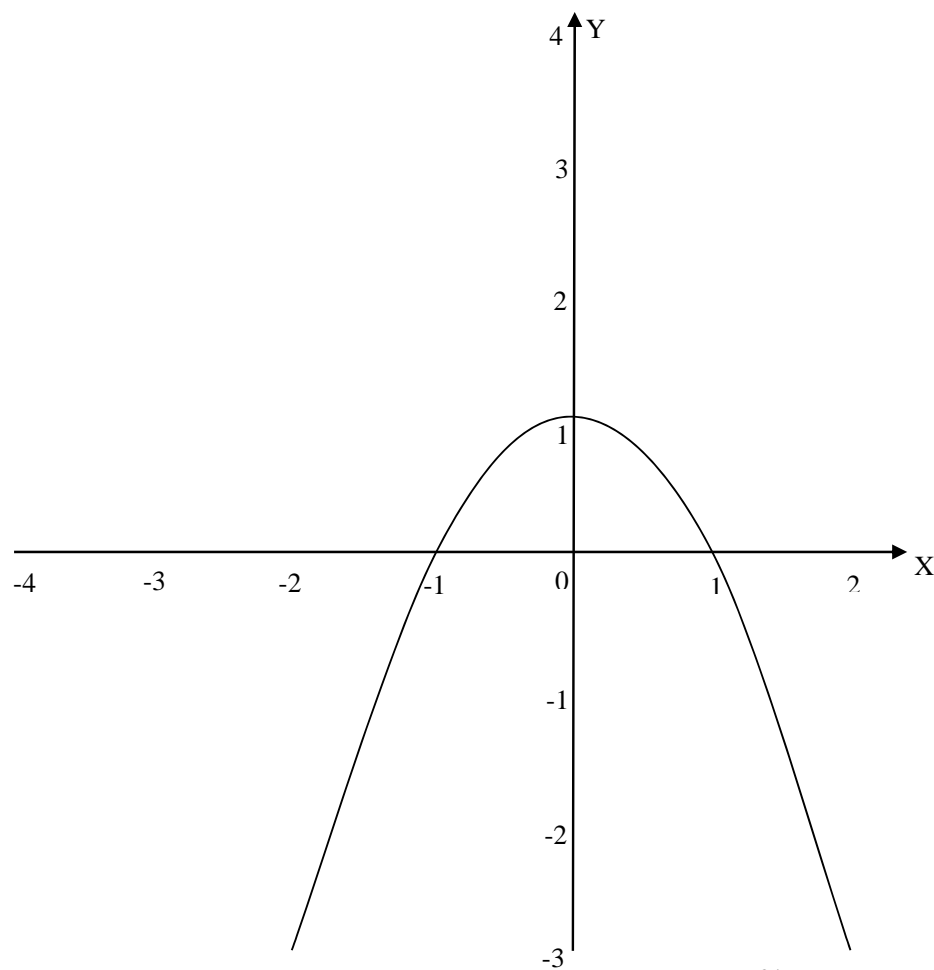
Exercise 2

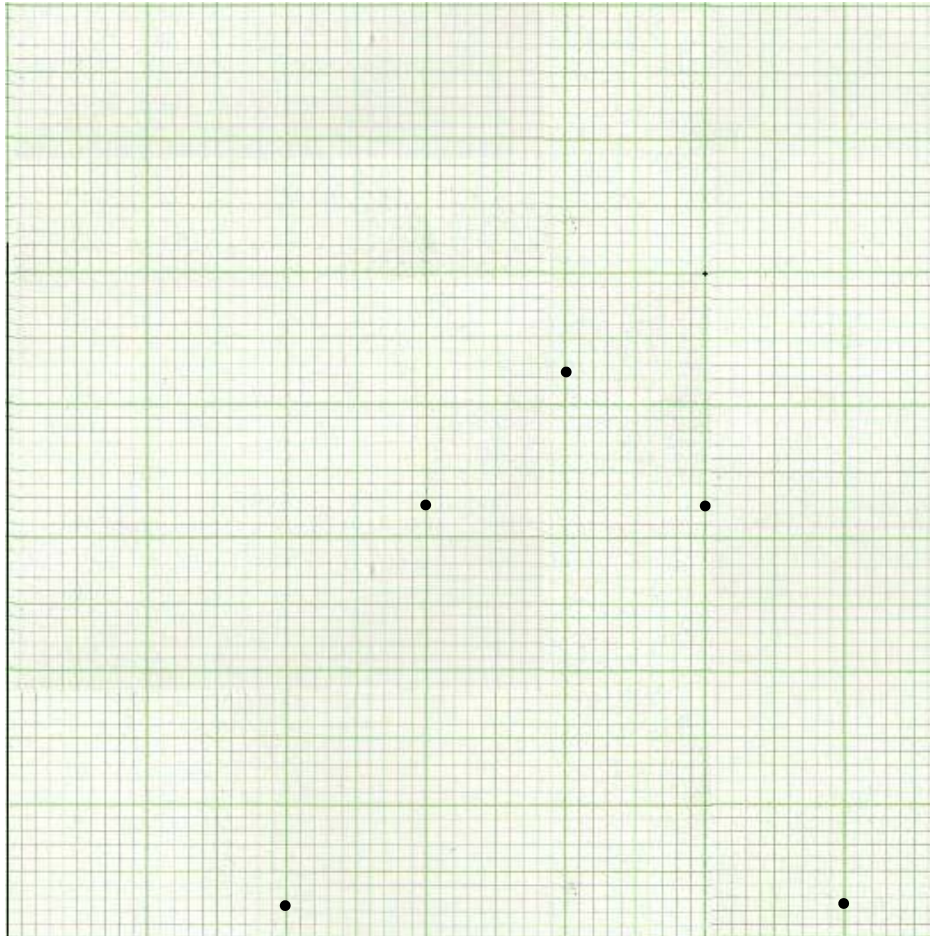
1. In answering this question, you needed to draw a graph using table of values as shown below:

The $f(x) = -x^2 + 1$.

X	-2	-1	0	1	2
Y	-3	0	1	0	-3

The graph will appear as shown below. You should note that since the coefficient of x^2 is negative, the graph is a maximum, turning downwards as shown below:

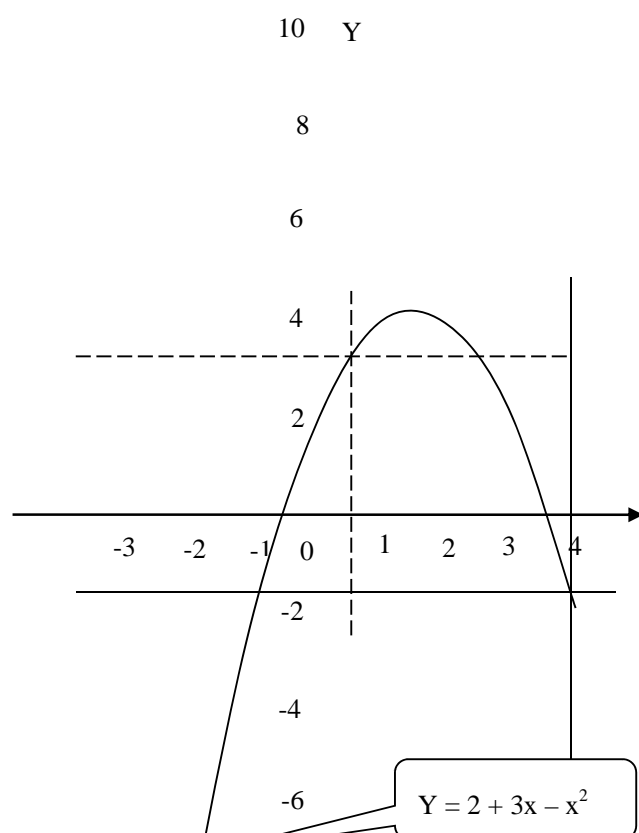


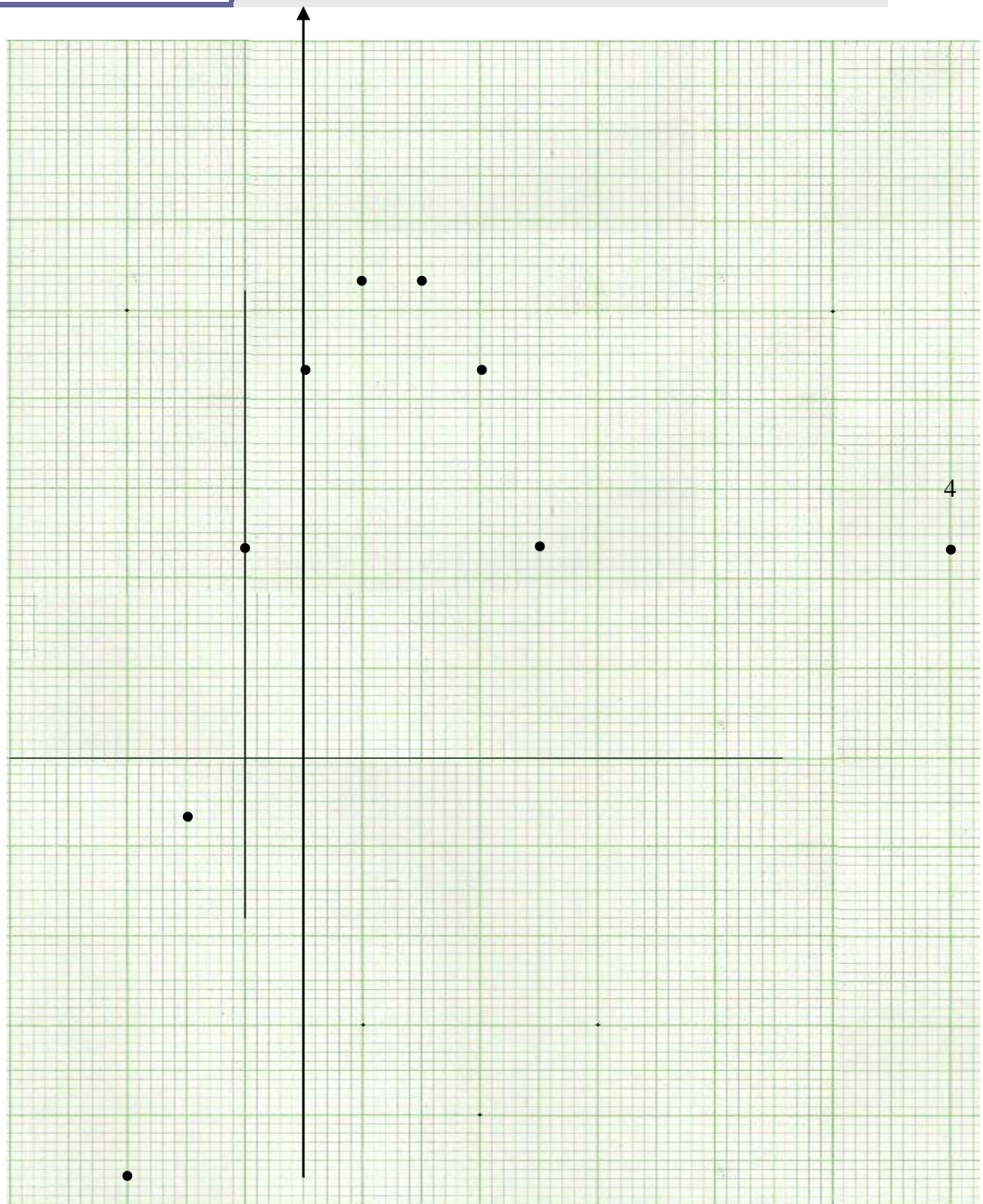


2. First you should have made the table of values to assist you draw the graph as shown below

The table for the graph $y = 2 + 3x - x^2$

X	-3	-2	-1	0	1	2	3	4
y	-16	-8	-2	2	4	4	2	-2





To use the graph, we have to find the values at :

(a) $x = 0.5$

We draw the line $x = 0.5$ to meet the curve. It meets the curve at $y \approx 3$

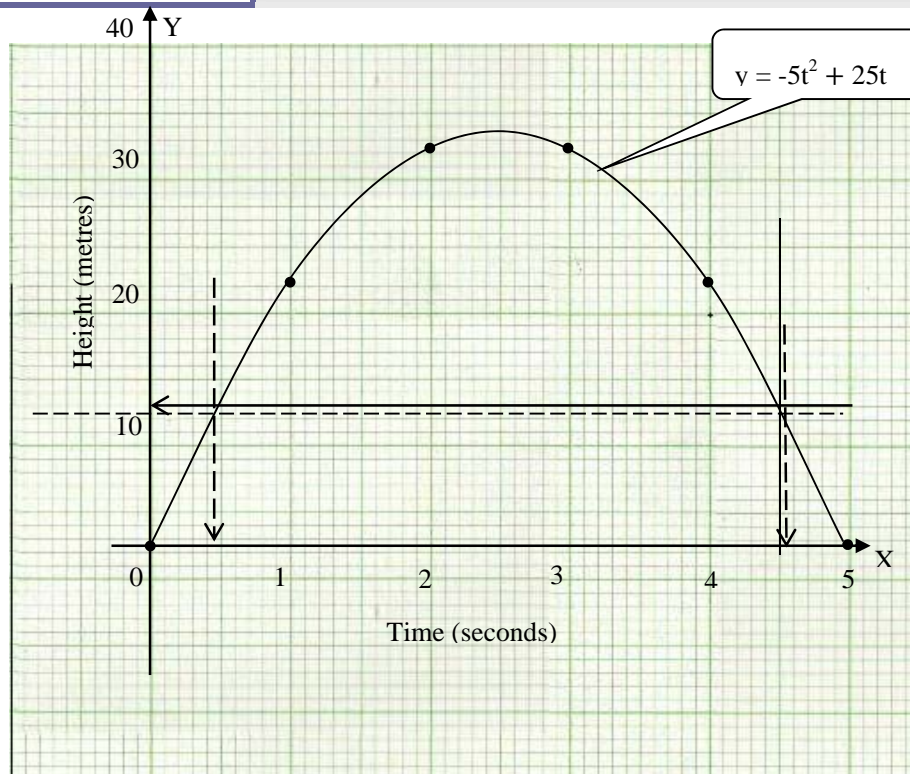
(b) $y = -1.5$

We draw the line $y = -1.5$ to meet the curve. It meets the curve at $x \approx -1.2$

3. First you should have made the table of values to assist you in drawing the graph. Then use the graph to answer the question asked as shown below

t	0	1	2	3	4	5
h	0	20	30	30	20	0

The graph is as shown below:



- (a) Estimation of time when the height is 10 m above the ground is done by drawing a line from the mark of 10 m on the vertical axis meeting the curve as shown above.

When the stone is 10 metres above the ground, time will be represented by t_1 and t_2 . Therefore, $t_1 \approx 0.5$ seconds and $t_2 \approx 4.6$ seconds

- (b) Estimation of the height is done in the similar manner. Remember that you should only use the graph. No calculations are necessary.

NB that the answers are estimations or rather approximations only. Hence the symbol being used: \approx

When $t = 4.5$, we draw a line from that point at $t = 4.5$ upwards to meet the curve. Then from the meeting point, we draw another to meet the height axis and then read out the value as shown above.

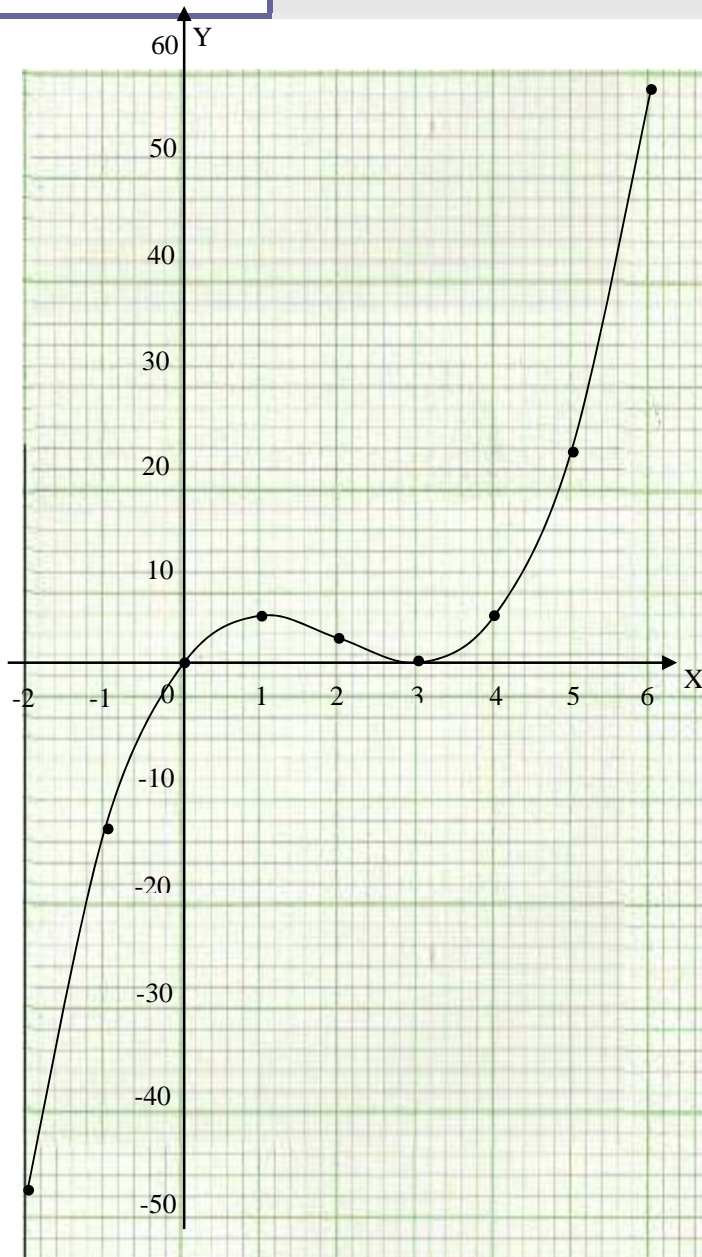
At time, $t = 4.5$, $H \approx 11$ metres.

- (c) The estimation for the time when the stone hits the ground is considered by marking the point at which the curve meets the time axis. When the stone hits the ground, $t = 5$ seconds.

4. To draw this graph, it was necessary for you to make the table of values first as shown.

It is necessary to make the table of values first as shown below:

X	-2	-1	0	1	2	3	4	5	6
Y	-50	-16	0	4	2	0	4	20	54



The answers to the part questions in this question are to be found by just considering the graph.

(a) Values of x for which $x(x - 3)^2 = 0$:

Check for the points at which the curve cuts the X-axis:

$(0, 0)$ and $(3, 0)$

Therefore, $x = 0$ and $x = 3$.

(b) Coordinates of the maximum point:

The point is (1, 4)

(c) Coordinates of the minimum point:

The point is (3, 0)

(d) The range of the function of y:

Here we consider the values of y given in the table for each value of x;

Range of $y = \{-50 \leq y \leq 54, x \in \mathbb{Z}\}$

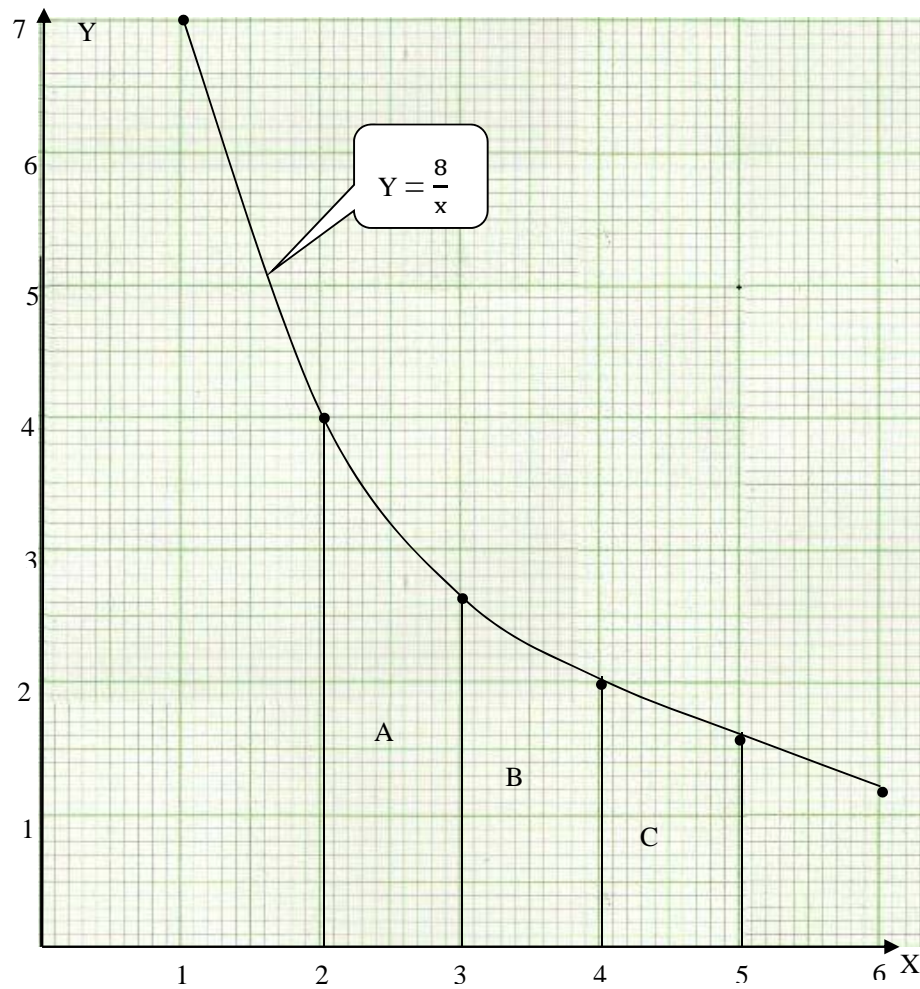
Exercise 3

1. $y = \frac{8}{x}$

The table of values

X	1	2	3	4	5	6
Y	8	4	$2\frac{2}{3}$	2	$1\frac{3}{5}$	$1\frac{1}{3}$

The graph



For area A, $a = 4$, $b = \frac{8}{3}$, $h = 1$

$$A = \frac{(a+b)h}{2}$$

$$A = \frac{\left(4 + \frac{8}{3}\right)1}{2} = \frac{\frac{12+8}{3}}{2} = \frac{20}{3 \times 2} = \frac{20}{6} = 3\frac{2}{6} \text{ square units.}$$

For area B, $a = \frac{8}{3}$, $b = 2$, $h = 1$

$$A = \frac{(a+b)h}{2}$$

$$A = \frac{\left(\frac{8}{3}+2\right)1}{2} = \frac{\frac{8+6}{3}}{2} = \frac{14}{3 \times 2} = \frac{14}{6} = 2\frac{2}{6} \text{ square units}$$

For area C, $a = 2$, $b = \frac{8}{5}$, $h = 1$

$$A = \frac{(a+b)h}{2}$$

$$A = \frac{\left(2+\frac{8}{5}\right)1}{2} = \frac{\frac{10+8}{5}}{2} = \frac{18}{2 \times 5} = \frac{18}{10} = 1\frac{8}{10}$$

Area for the region under the curve = area A + area B + area C

$$\text{Area} = \frac{20}{6} + \frac{14}{6} + \frac{18}{10}$$

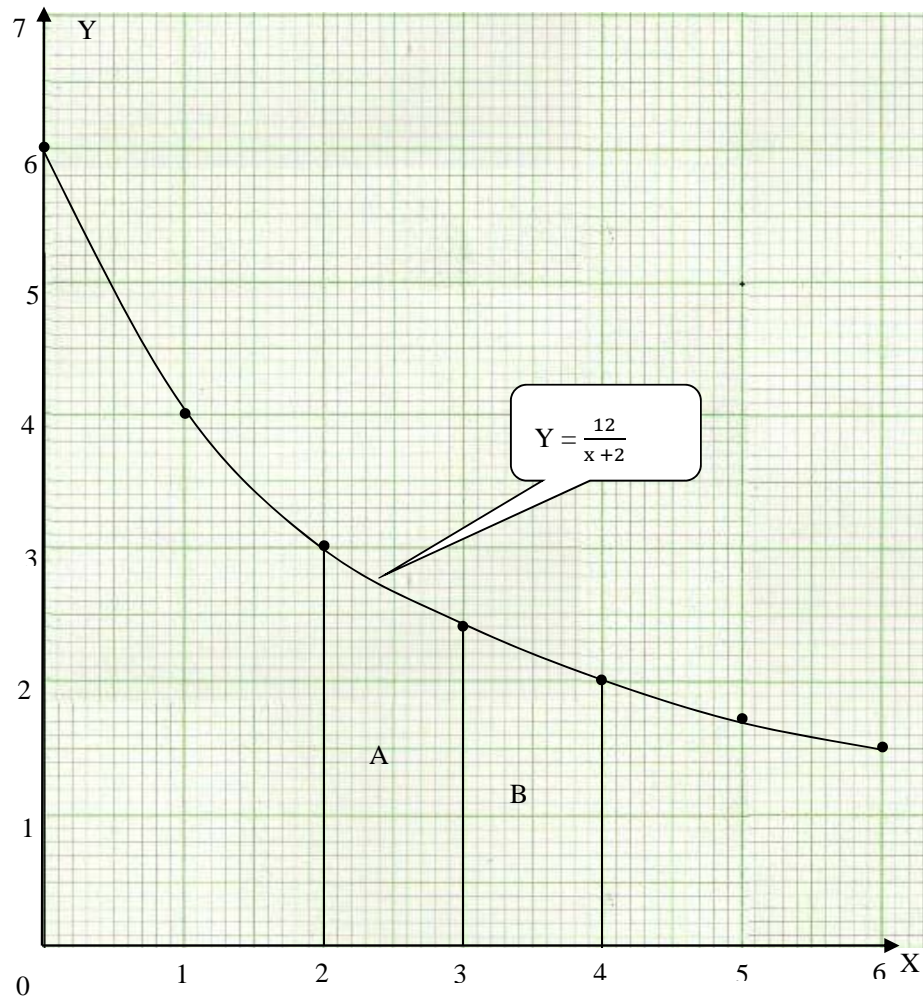
$$\text{Area} = \frac{100+70+54}{30} = \frac{224}{30} = 7\frac{14}{30} \text{ square units.}$$

$$2. y = \frac{12}{x+2}$$

The table of values

X	0	1	2	3	4	5	6
Y	6	4	3	$2\frac{2}{5}$	2	$1\frac{5}{7}$	$1\frac{1}{2}$

The graph



For area A, $a = 3$, $b = \frac{12}{5}$, $h = 1$

$$A = \frac{(a+b)h}{2}$$

$$A = \frac{\left(3 + \frac{12}{5}\right)1}{2} = \frac{\frac{15+12}{5}}{2} = \frac{\frac{27}{5}}{2} = \frac{27}{5 \times 2} = \frac{27}{10} = 2 \frac{7}{10}$$

square units

For area B, $a = \frac{12}{5}$, $b = 2$, $h = 1$

$$A = \frac{(a+b)h}{2}$$

$$A = \frac{\left(\frac{12}{5} + 2\right)1}{2} = \frac{\frac{12+10}{5}}{2} = \frac{\frac{22}{5}}{2} = \frac{22}{5 \times 2} = \frac{22}{10} = 2 \frac{2}{10}$$

square units

Area under the curve = area A + area B

$$\text{Area under curve} = \frac{27}{10} + \frac{22}{10} = \frac{27+22}{10} = \frac{49}{10} = 4 \frac{9}{10} \text{ square units.}$$

Assessment



You now can answer the tutor marked assignment below. You are expected to be honest to yourself as you attempt these questions. This will help you determine how much you have acquired in terms of knowledge and skills. After answering these questions, please send the answers to a centre near you for marking. After marking it will be sent back to you for attention and to know your performance. This assessment consists of contents from unit 1, unit 2 and unit 3. The questions in this assignment have been taken from general mathematics-revision and practice.

There are five (5) questions in this assessment. You are to attempt all. Where necessary, show your working as you arrive at the solution.

1. Estimate the value of $\frac{112.2 \times 75.9}{6.9 \times 5.1}$ by rounding off the numbers correct to 1 significant figure.
2. Given that $L = 2\sqrt{\frac{a}{k}}$, find the value of L in standard form when $a = 4.5 \times 10^{12}$ and $k = 5 \times 10^7$
3. Given the function $h(x) = x^2 + 1$, find (i) $h(2)$ (ii) $h(-3)$

4. Given that $y = \frac{9}{x}$, complete the following table of values, stating the values where appropriate to two decimal places.

X	1	2	3	4	5	6	7	8	9
y	9		3				1.29	1.13	

- (a) Draw the graph of $y = \frac{9}{x}$ for $1 \leq x \leq 9$. Use a scale of 1 cm to 1 unit on both axes.
- (b) Using your graph and showing your method clearly, obtain an approximate solution of the equation $\frac{9}{x} = 2.4$
- (c) Find the approximate area under the curve between $x = 1$ and $x = 2$.
5. A rectangle has a perimeter of 14 cm and length x cm. Show that the width of the rectangle is $(7 - x)$ cm and hence that the area A of the rectangle is given by the formula $A = x(7 - x)$. Draw the graph using the scale of 2 cm to 1 unit on the X axis and 1 cm to 1 unit for values of x from 0 to 7. From the graph find
- The area of the rectangle when $x = 2.25$ cm
 - The dimensions of the rectangle when its area is 9 cm^2
 - The maximum area of the rectangle
 - The length and width of the rectangle corresponding to the maximum area.

Source: Rayner, D (1988), General Mathematics-Revision and Practice

Answers to the Assessment



1. To estimate the value, the numbers are rounded off to make the calculations easier.

$$\frac{112.2 \times 73.9}{6.9 \times 5.1} \approx \frac{100 \times 70}{7 \times 5} \approx \frac{7000}{35} \approx 200$$

2. $L = 2\sqrt{\frac{a}{k}}$

We know that $a = 4.5 \times 10^{12}$ and $k = 5 \times 10^7$

By substituting these values:

$$\begin{aligned} L &= 2\sqrt{\frac{4.5 \times 10^{12}}{5 \times 10^7}} \\ L &= 2\sqrt{\frac{4.5 \times 10^{12-7}}{5}} \\ L &= 2\sqrt{0.9 \times 10^5} \\ L &= 2\sqrt{90000} \\ L &= 2 \times 300 \\ L &= 600 \\ L &= 6 \times 10^2 \end{aligned}$$

3. We are given that $h(x) = x^2 + 1$

(i) $h(2)$
 $h(2) = (2)^2 + 1 = 4 + 1 = 5$

(ii) $h(-3)$
 $h(-3) = (-3)^2 + 1 = 9 + 1 = 10$

4. Given the function: $y = \frac{9}{x}$

(a) To draw the graph, we need to find the missing values in the table. This is done by substitute the values of x:

When $x = 2$, $y = \frac{9}{2} = 4.5$.

Using the above, the others are: (4, 2.25), (5, 1.8), (6, 1.6), (9, 1)

Completed Table

X	1	2	3	4	5	6	7	8	9
Y	9	4.5	3	2.25	1.8	1.6	1.29	1.13	1

Table 1: Values of x and y for $y = \frac{9}{x}$

The graph of $y = \frac{9}{x}$

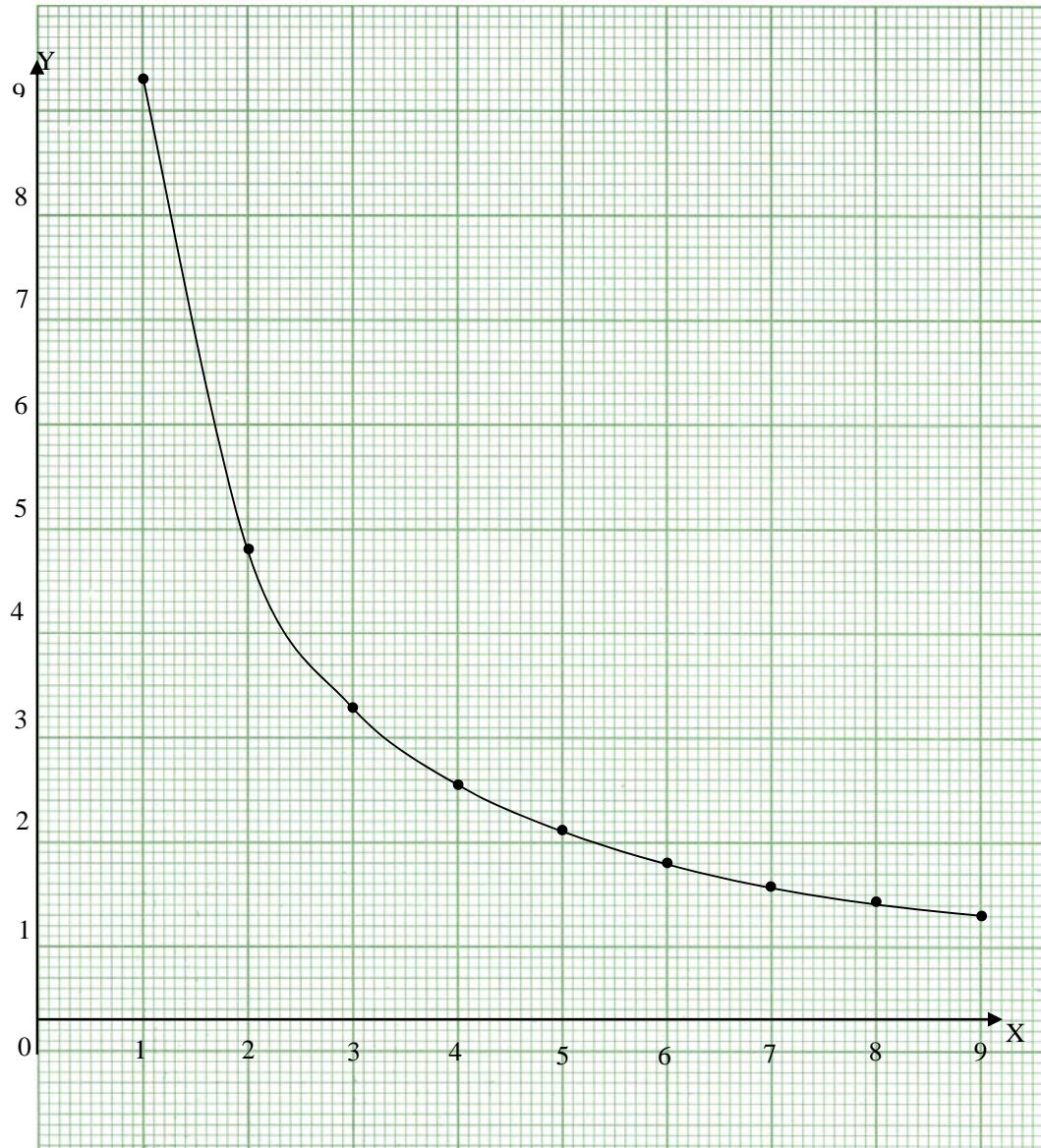
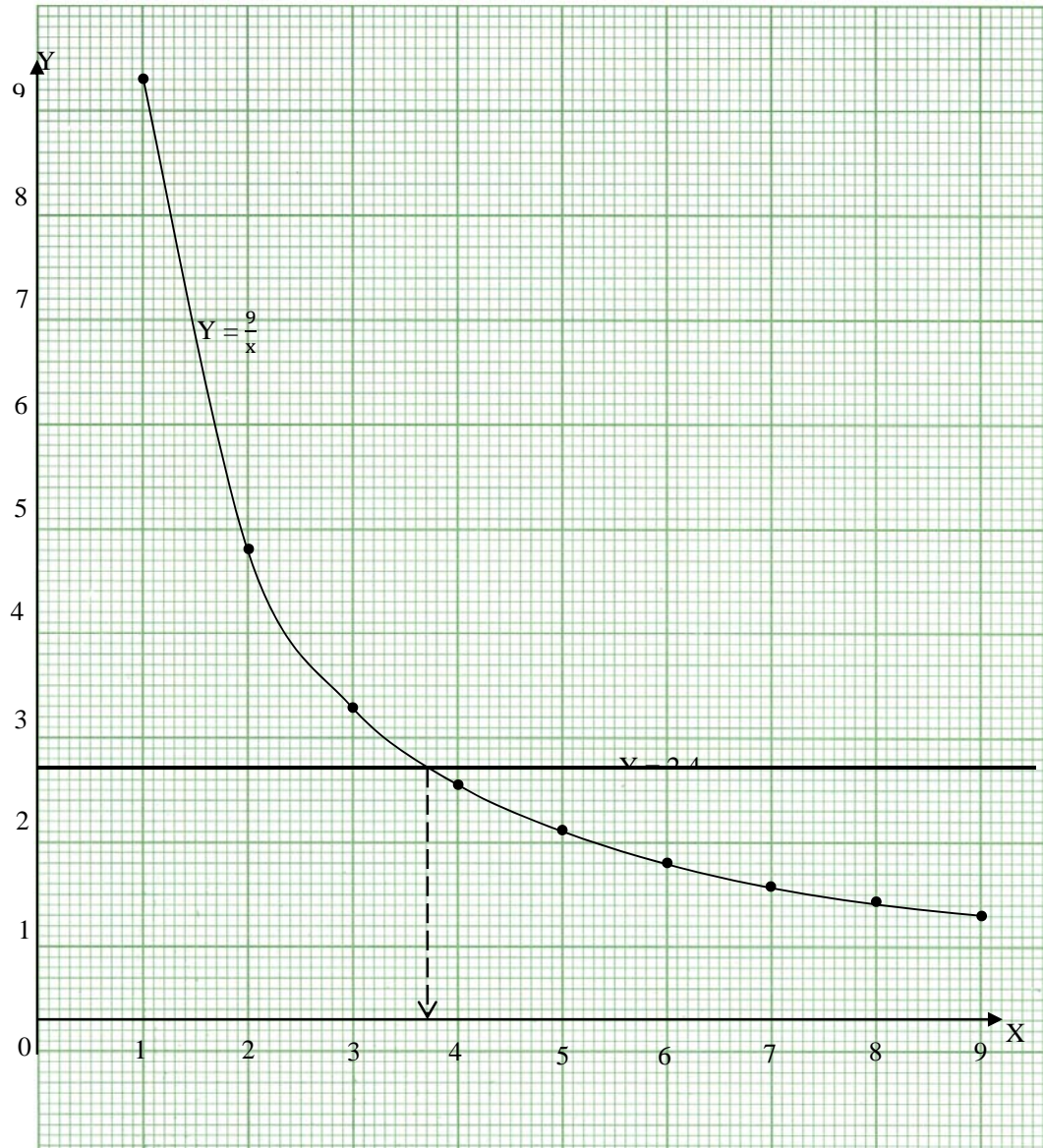


Figure: Graph of $y = \frac{9}{x}$

(b) Approximate solution of the equation $\frac{9}{x} = 2.4$.

Since $y = \frac{9}{x}$, it means that $y = 2.4$. Therefore, we draw a straight line through $y = 2.4$ and which cuts the graph at some point as shown in the graph below.



Therefore, the solution is: $x = 3.7$

(c) Approximate area under the graph.

The area under discussion is between $x = 1$ and $x = 2$ as shown as Area A below:

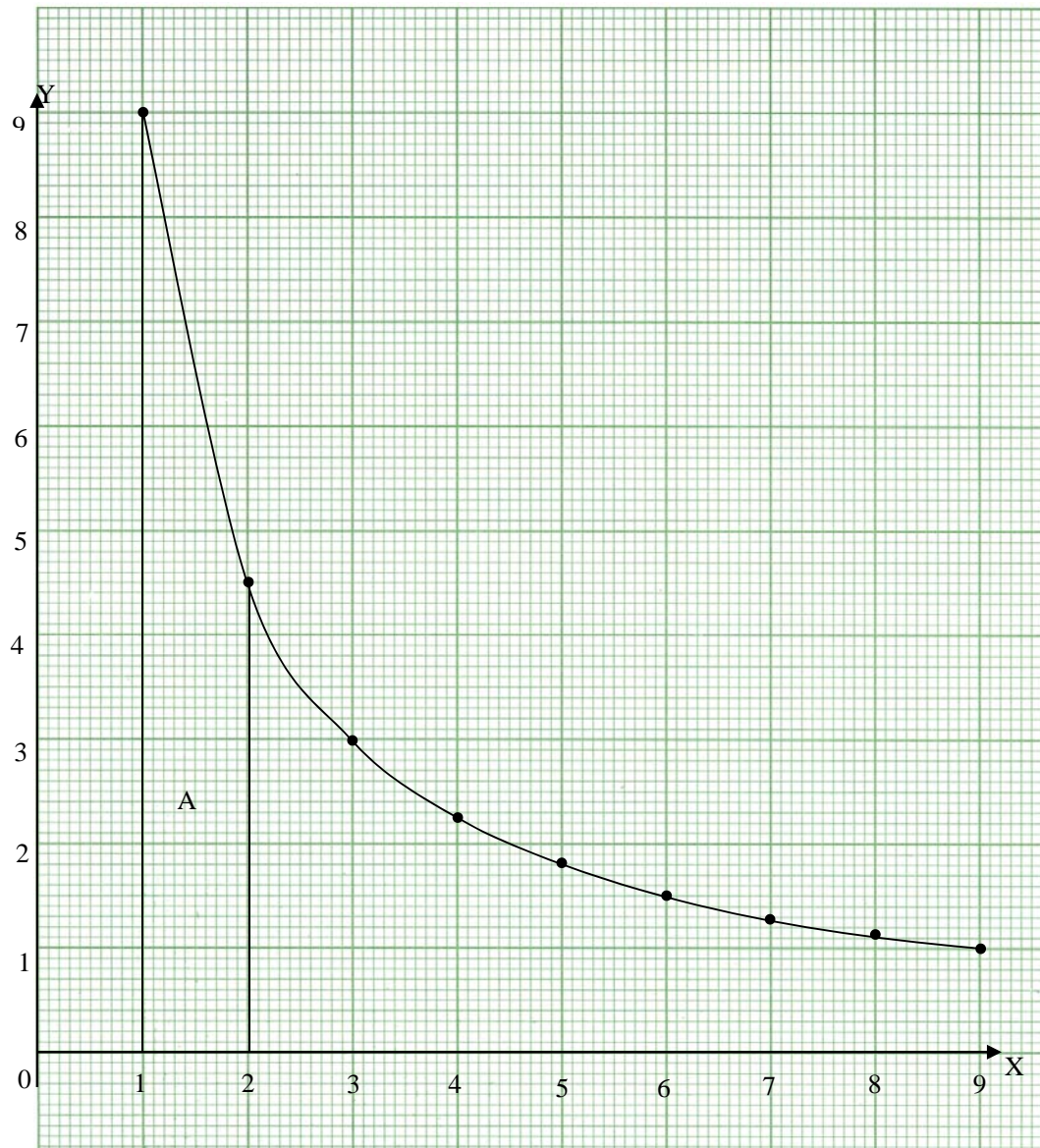


Figure 3: Graph of $y = \frac{9}{x}$

The formula for finding the area of a trapezium is given by the following: $A = \frac{1}{2}(a + b)h$

We know that side $a = 9$ units and side $b = 4.5$ units

For area $A = \frac{1}{2}(9 + 4.5) \times 1 = \frac{1}{2}(13.5) = 6.5$ sq. units

Total area under the curve \approx area A

Total area under the curve ≈ 6.5

Therefore, total area ≈ 6.5 square units.

NB: Though the work on graph is shown as three different graphs, it can be done on one graph paper and drawing. This has been done for guidance sake. Otherwise, you are required to show all the work on one graph paper.

5. Perimeter = 14, length = x .

Using the perimeter, we can find the value of the breadth or width.

$$P = 2(l + b)$$

$$14 = 2(x + b)$$

$$7 = x + b$$

$$b = 7 - x$$

Therefore, width = $7 - x$

The area of a rectangle is given by: $A = lb$

Using the above values:

$$A = lb$$

$$A = x(7 - x)$$

In this statement, $A = x(7 - x)$, area(A) is a function of x . This means that the graph drawn will reflect the area of the rectangle with respect to the value of x .

Table of values

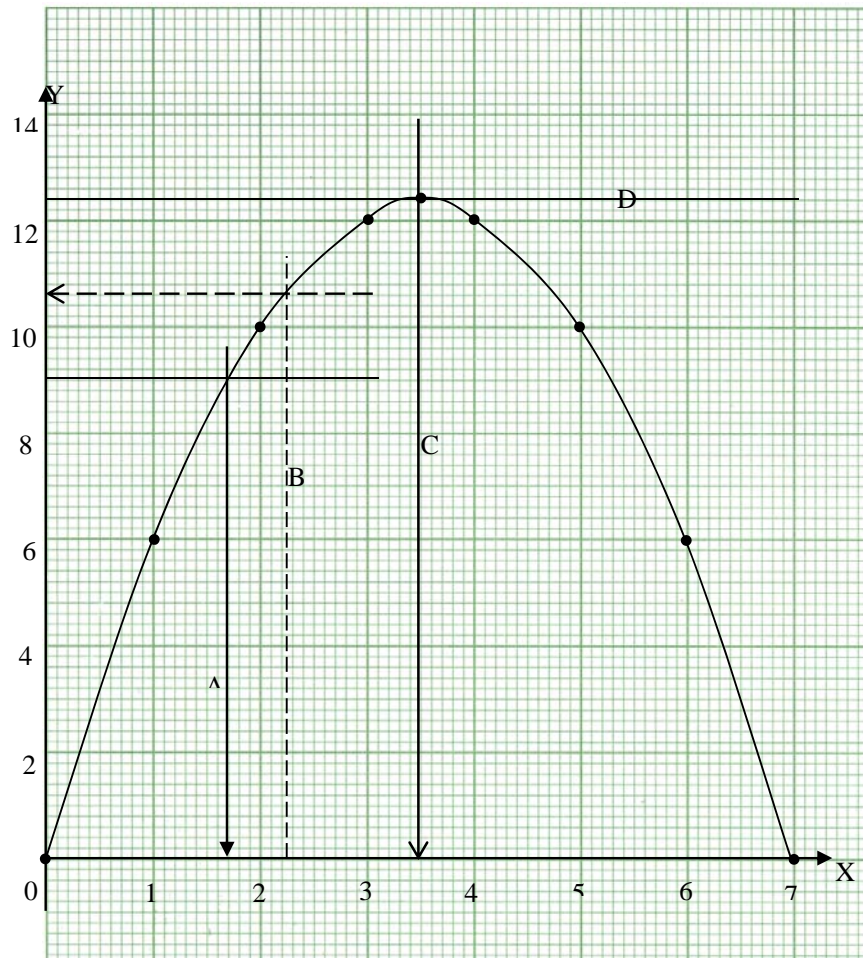
The values of A are found by substituting the values of x in the formula.

$$\text{When } x = 0, A = 0(7 - 0) = 0$$

Using the above calculations, the values are calculated and the completed table is as follows:

X	0	1	2	3	4	5	6	7
A	0	6	10	12	12	10	6	0

As stated in question 4, all the work will be shown on one graph. Each part of the question is answered on the same graph paper. There will be only one drawing of the graph but all related questions are to use the same graph.

The graph of $A = x(7 - x)$ 

- (a) Area ≈ 10.6 when $x = 2.25$ (follow Line B)
- (b) $X \approx 1.7$ and width ≈ 5.3 , $A = 9$ (follow Line A)
- (c) Maximum area $\approx 12.4 \text{ cm}^2$ (follow line C and D)
- (d) At maximum area, $x \approx 3.5$ and width ≈ 3.5

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Unit 4

Quadratic Equations

Introduction

Welcome to unit 4 of mathematics 11. You have so far studied three units in grade 11. You learnt about approximations and scientific notations in the first unit. This will prove to be beneficial since some of the solutions given in this unit are approximations. In the second unit you learnt about relations and functions. This unit introduced to you relations and functions in general. The concepts on functions which you learnt can be applied to quadratic equations. In the previous unit, unit three, you considered graphs of polynomials. You learnt how to draw and interpret graphs of linear, quadratic and cubic functions. You also learned how to calculate the gradient of a curve at a given point. You further learned how to estimate the area under a curve. The skills you acquired drawing graphs will be useful since one method of solving quadratic equations requires drawing graphs.

In this unit you will learn about quadratic equations. You learned about quadratic expressions in your mathematics 10 course. There is no big difference between quadratic expressions and quadratic equations. The difference is that quadratic equations have an equal sign while quadratic expressions do not. A more detailed explanation on the difference between the two will be given as the discussion progresses.

You have already learnt how to factorise quadratic expressions. Be assured that this will be very useful in the study of this unit since at some point you will be required to factorise as you solve quadratic equations. So what you learnt earlier will act as part of the foundation of what you will learn in this unit.

So the knowledge acquired in grade 10 and expanded on in the previous units is essential if you are to understand the material in this unit with ease. You are, therefore, encouraged to revise your grade 10 work on factorization of quadratic expressions.

In this unit we are looking at solving quadratic equations. We are going to look at four ways of solving quadratic equations. These methods include factorisation, completing the square, quadratic formula and graphical methods.

Knowledge of quadratic functions is very helpful since it can be applied in our everyday situations. For example in construction of overhead bridges the knowledge of quadratic equations is applied since the design of the bridges is similar to that of a parabola (the shape of the graph of a quadratic function as explained in unit 3). Quadratic equations can also be applied in determining the number of floor tiles needed for a particular room.

There are three topics in this unit. The first topic introduces quadratic equations and explains what they are. It also discusses the three algebraic methods of solving quadratic equations. The second topic discusses the graphical method of solving quadratic equations. The third topic is on the application of quadratic equations.

Upon completion of this unit you will be able to:



Outcomes

- *Solve* quadratic equations algebraically.
- *Solve* quadratic equations graphically.
- *Apply* quadratic equations to solve mathematical problems.

Timeframe



How long?

We estimate that to complete studying this unit you will need between 18 and 22 hours. This time includes the time you will spend doing the self-marked activities. The first topic will take about 9 hours while the second and third will take about 7 and 5 hours respectively. If you do not finish within this estimated time do not worry since we do not all learn at the same pace.

You are encouraged to spend not more than 3 hours on the first self-marked exercise and not more than 2 hours on either the second or third self-marked exercises.

Learning resources



Tip

As in the other units that you have studied in this course you will need the following resources to be able to study the unit with minimal difficulties:

- Ruler
- Pencil
- Graph Paper
- Eraser
- Calculator

Teaching and Learning Approaches



Tip

In this unit we have used three teaching and learning methods. These are:

- **Conceptual:** This method will help you learn facts, rules, formulas and procedures in mathematics. We have illustrated how facts and rules can be applied and, where possible, they have been defined. Formulas and procedures have been explained and examples of how they can be applied have been given.

- **Problem-solving:** This method will help you to apply mathematics to real life situations and also enable you to develop problem-solving strategies. The unit will help you develop skills which will assist you to reason on given situations.
- **Skills:** This method will help you to practice using the facts, rules, formulas and procedures you are learning. In this unit there are a number of activities that will give you a chance to practice what you are learning.

This unit, as all the other units in this course, will also be interactive. There will be dialogue between us and you will be responding to in-text questions and doing activities using blank spaces. These spaces can also be used for making your own notes. Throughout the unit you will be able to assess your progress and understanding by doing the activities. After studying through each topic you will be required to assess yourself by working out the exercises and marking your own work using the feedback provided immediately after each exercise. You are encouraged to do the exercises.



Terminology

Coefficient:	A number which is placed before another quantity and which multiplies that quantity.
Constant:	A number or quantity that does not vary.
Equation:	An open statement formed by two expressions separated by an equal sign.
Expression:	A mathematical statement such as $ax + 2x$ or $3x - 2a + 4b$.
Formula:	An equation stating the relationship among quantities that can be represented by variables.
Integer:	Any number in the set $\{ \dots, -2, -1, 0, 1, 2, \dots \}$.
Perfect Square:	Quadratic expression with two equal factors.
Real Number:	A number that can be expressed as $\frac{a}{b}$, where a and b are integers and $b \neq 0$.
Root:	Solution of a quadratic equation.
Rounding:	The process of replacing a number by an approximate number.
Term:	The product of one or more numerical factors and variable factors.

Variable: A letter or symbol used to represent a number.

Topic 1 Algebraic Methods

As indicated earlier, this is the first of three topics in this unit on quadratic equations. In this unit you will learn how to solve quadratic equations algebraically by factorization, completing the square and quadratic formula methods. Knowledge of these methods will be very useful because you will be able to apply it in solving many mathematical problems which lead to quadratic equations. You are encouraged to master all of them so that you can decide which one to use at a particular time.

This topic is dedicated to addressing the first of the three unit outcomes, namely:

- Solve quadratic equation algebraically.

This outcome has been broken down into three objectives and these are that upon completion of this topic you will be able to:



Objectives

- *Solve* quadratic equations by factorisation.
- *Solve* quadratic equations by completing the square.
- *Solve* quadratic equations by quadratic formula.

As we start addressing the first of the above objectives we need to explain what quadratic equations are.

What Are Quadratic Equations?

In your study of mathematics you must have come across quadratic expressions in a unit on factorization in your mathematics 10 course, but what are quadratic equations?

You must recall that a quadratic expression consists of terms with 2 as the highest power of the variables. The general form of a quadratic expression is $ax^2 + bx + c$, where $a \neq 0$. You should be in a position to identify a quadratic expression.

Now do this simple activity. In this activity you have to use your previously acquired knowledge on quadratic expressions.



Activity 1

Look at the following expressions:

(i) $x^2 - 7x + 10$

(ii) $x + 4$

(iii) $x^2 + 18x + 60$

(iv) $x^3 - x^2 + x + 6$

Which of these equations are quadratic expressions? Write your answer in the space that follows.

Did you indeed find this activity as easy as I said it would be? Good! You can now check whether you arrived at the same answer as mine



Feedback

Expressions (i) and (iii) are the only quadratic expressions amongst the four. As you can see the highest power of x in the expressions $x^2 - 7x + 10$ and $x^2 + 18x + 60$ is 2. The highest power of $x + 4$ is 1 and that of $x^3 - x^2 + x + 6$ is 3. Therefore the expressions in (ii) and (iv) are not quadratic expressions.

The following are also examples of quadratic expressions in one variable x :

(i) $x^2 + 2x - 4$

(ii) $2x^2 - x - 4$

The highest power of the expressions is 2. Thus, they are quadratic expressions.

The variable x is not the only one that can be used in a quadratic equation. Even other letters can be used as in the following expressions:

(i) $P^2 - 3p + 4$

(ii) $m^2 + 2m + 2$

(iii) $k^2 - 6k + 4$

In (i), (ii) and (iii) the letters p , m and k respectively have been used showing that any letter can be used. This is so because the expressions may represent different things. For instance the expression $t^2 - 5t + 6$ may represent a distance and the variable t may represent time from the start.

So far we have looked at examples of quadratic expressions. You have seen that quadratic expressions can have any variable and that the highest power of variables is 2. Remember we asked a question 'what are quadratic equations?' Look at the following:

(i) $x^2 - 6x + 2$

(ii) $x^2 - 6x + 2 = 0$

The first one is a quadratic expression while the second one is a quadratic equation. An equation has an equal sign while an expression does not.

From the quadratic expressions in activity 1 we can come up with quadratic equations. From the expression $x^2 - 7x + 10$ we can come up with the equation $x^2 - 7x + 10 = 0$. An equal sign has now been introduced. Similarly from the expression $x^2 + 18x + 60$ we can come up with the equation $x^2 + 18x + 60 = 0$.

2 is the highest power

From the explanation above you can see that in both quadratic expressions and quadratic equations the highest power of variables is 2. You can also see that equations have an equal sign while expressions do not.

Now that we have established the difference between quadratic expressions and quadratic equations let us look at a general definition of a quadratic equation.

An equation of the form $ax^2 + bx + c = 0$, where **a**, **b** and **c** are constants and $a \neq 0$, is called a **quadratic equation** in one variable x . A variable is a letter or symbol used to represent a number. In the equation $ax^2 + bx + c = 0$, x is the only variable. A constant is a number or quantity that does not vary. So the variable x may take up different values but **a**, **b** and **c** remain the same for a given equation.

At this point we find it appropriate to explain what a coefficient is. Briefly, a coefficient is a number that is placed before a variable. In the general equation above **a** and **b** are coefficients. From the definition of a quadratic equation you can see that:

1. **a** is the coefficient of x^2 ,
2. **b** is the coefficient of x ,
3. **c** is a constant.
4. **a** can be either positive or negative but not zero, and
5. **b** and **c** can take any value including zero.

Here is an example to help you be in a better position to know the values of **a**, **b** and **c** in any given quadratic equation.

Example 1

In the equation $2x^2 - x + 3 = 0$, **a** is 2, **b** is -1 and **c** is 3.

In the equation $x^2 + 2x - 4 = 0$, **a** is 1, **b** is 2 and **c** is -4.

In the equation $-3x^2 + 4x = 0$, **a** is -3, **b** is 4 and **c** is 0.

In the equation $3 - 5y - 2y^2 = 0$, **a** is -2, **b** is -5 and **c** is 3.

In the equation $y^2 - 4 = 0$, **a** is 1 **b** is 0 and **c** is -4.

In the equation $x^2 = 0$, **a** is 1, **b** is 0 and **c** is 0.

In the equation $-x^2 + 9 = 0$, **a** is -1 **b** is 0 and **c** is 9.

You must have noticed that in the above examples the equations have different number of terms. This has to do with the values of **b** and **c**. If **b** and **c** are equal to zero then there will only be one term as in the case of $x^2 = 0$. If **c** is zero then we will have two terms as in the case of $-3x^2 + 4x = 0$ and if **b** is zero we will have two terms as in $-x^2 + 9 = 0$.

You should now do the following activity. The activity will test your comprehension of the general definition of a quadratic equation $ax^2 + bx + c = 0$.



Activity 2

State the values of **a**, **b** and **c** in the equation $2x^2 - x - 4$. Write your answer in the space below.

You can now go ahead and check the feedback.



Feedback

*The value of **a** is 2 since the coefficient of x^2 is 2.
The value of **b** is -1 since the coefficient of x is -1 .
The value of **c** is -4 since the coefficient -4 is the constant in the equation.*

Now that we have discussed the definition of a quadratic equation let us focus on the methods of solving these equations.

Quadratic equations, as mentioned earlier can be solved algebraically by three methods. These methods are **factorisation**, **completing the square** and by applying the **quadratic formula**. In this unit you will first learn how to solve quadratic equations by factorisation. You will then learn how to solve the equations by completing the square. You will also learn about the third algebraic method of solving quadratic equations which involves applying the quadratic formula. Now let us look at the factorisation method.

Solving Quadratic Equations by Factorisation

You learned about factorization of quadratic expressions in your mathematics 10. The knowledge and skills of factorising will be helpful. In solving a quadratic equation, you need to do more factorising. Apart

from factorising you will also have to work out the values of the variable in the factors. Since the process of factorisation is essential in this method it is important that you know it. To assess how well you know it do the following activity.

The activity involves factorising a quadratic expression.



Activity 3

Factorise the expression $x^2 + 5x + 6$ and write your answer in the space below.



Feedback

$$x^2 + 5x + 6$$

From your previously acquired knowledge you will realize that the coefficient of x^2 is 1 and the constant is 6. The product of 1 and 6 is 6. So we have to get two numbers such that when we multiply them we obtain 6. The pairs of numbers that results in 6 after multiplication are 2 and 3, -2 and -3, 1 and 6 as well as -1 and -6.

Also the coefficient of x is 5. From the pairs of numbers we have just listed we have to choose one such that if we add the two numbers in the pair we should get 5. In this case 2 and 3 satisfy this condition because $2 + 3 = 5$ or $5 = 2 + 3$. So we use 2 and 3 to rewrite $5x$ as $2x + 3x$. Thus

$$x^2 + 5x + 6$$

replacing $5x$ with $2x + 3x$ we get

$$x^2 + 2x + 3x + 6$$

$5x$ is replaced with
 $2x + 3x$ because
 $5x = 2x + 3x$

We group the terms and obtain:

$$x(x + 2) + 3(x + 2)$$

$$x^2 + 2x = x(x + 2)$$

$$\text{and } 3x + 6 = 3(x + 2)$$

$(x + 2)$ is common and the other factor is obtained from the terms outside the brackets. These terms are x and $+ 3$. Thus we have:

$$(x + 2)(x + 3)$$

We have briefly revised factorisation of quadratic expressions. We are now going to outline, in general, the steps for factorising a quadratic expression. The method involves four steps and these steps will be explained using the general form for a quadratic expression $ax^2 + bx + c$. To solve $ax^2 + bx + c$ use these four steps:

Step 1: find the product of **a** and **c**. (A product of two numbers is the number obtained when the two numbers are multiplied together.)

Step 2: find the factors of the product of **a** and **c** whose sum is **b**. (A sum of two numbers is the number obtained when the two numbers are added together.)

Step 3: express the term **bx** as a sum of terms using the factors in step 2.

Step 4: factorize by grouping terms with common factors.

You now know the steps used to factorize quadratic expressions; our next task is to look at an example of how you can solve a quadratic equation by factorization. In the example the coefficient of x^2 is not 1 as was the case in activity 3.

Example 2

Solve the equation $2x^2 + 3x - 9 = 0$

Solution

In the equation $2x^2 + 3x - 9 = 0$; $a = 2$, $b = 3$ and $c = -9$. So $a \times c = -18$.

The factors of -18 whose sum is $b = 3$ are -3 and 6. Therefore $3x$ in the equation can be expressed as $-3x + 6x$ using -3 and 6.

Thus the equation $2x^2 + 3x - 9 = 0$ can be written as:
 $2x^2 - 3x + 6x - 9 = 0$

After grouping terms and factorizing we get:
 $x(2x - 3) + 3(2x - 3) = 0$.

Factorizing further we get $(2x - 3)(x + 3) = 0$.

Now since $(2x - 3)(x + 3) = 0$, either $2x - 3$ is equal to zero or $x + 3$ is equal to zero. So we get
 $2x - 3 = 0$ or $x + 3 = 0$

We now work out the values of x in both expressions:

We divide both sides by the coefficient of x

$$\begin{aligned} 2x &= 3 \text{ or } x = -3 \\ \frac{2x}{2} &= \frac{3}{2} \text{ or } x = -3 \\ x &= 1.5 \text{ or } x = -3 \end{aligned}$$

If you followed the example above closely it should be easy for you solve the equation by factorization. Do the following activity to see whether or not you have understood what we have been discussing.

**Activity 4**

Rewrite the equation $8x^2 - 4x - 4 = 0$ expressing $-4x$ as a sum of terms using factors of the product of 8 and -4. Write your answer in the space below.

That was easy. Was it not? You can now compare your answer with the feedback.



Feedback

$$8 \times -4 = -32$$

The factors of -32 whose sum is -4 are -8 and 4 . Therefore $-4x = -8x + 4x$.

Thus $8x^2 - 4x - 4 = 0$ can be expressed as $8x^2 - 8x + 4x - 4 = 0$

If you did not find this activity easy, you should first revise the steps we presented above and then try to do the activity again. If you still find it difficult, then you should discuss it with your tutor or other learners at the learning centre.

At this point let us look at another example on how to solve quadratic equations by factorization. This example is meant to consolidate the knowledge you have acquired so far on solving quadratic equations by factorization.

Example 3:

Let us solve the equation $2x^2 - 5x + 2 = 0$ by factorizing.

Solution:

$$2x^2 - 5x + 2 = 0$$

$$2x^2 - 4x - x + 2 = 0$$

$$2x(x - 2) - 1(x - 2) = 0$$

$$(2x - 1)(x - 2) = 0$$

$$(2x - 1) = 0 \text{ or } (x - 2) = 0$$

$$2x = 1 \text{ or } x = 2$$

$$x = \frac{1}{2} \text{ or } x = 2$$

$5x$ is expressed as $-4x - x$ using -4 and -1 . This is because $-4 + -1 = -5$ and $-4 \times -1 = 4$ as explained above

$2x - 1$ is formed from the terms outside the brackets and $x - 2$ is taken as it is from the stage just before. $x - 2$ is a factor.

Now that you have looked at several examples it should give you enough confidence to do the following activity to test your knowledge. All you need to successfully do the activity is apply what we have discussed so far.



Activity 5

State the values of **a**, **b** and **c** and the steps that must be taken to factorize $12x^2 + 2x - 4 = 0$ in the space below.



Feedback

$$a = 12, b = 2 \text{ and } c = -4$$

Step 1: find the product of a and c

$$12 \times -4 = -48$$

Step 2: find the factors of the product of a and c whose sum is b .

The factors of -48 whose sum is 2 are -6 and 8 .

Step 3: express the term $-48x$ as a sum of terms using the factors in step 2. Hence $12x^2 + 2x - 4 = 0$ becomes $12x^2 + -6x + 8x - 4 = 0$

Step 4: factorize by grouping terms with common factors.

$$\begin{aligned} 12x^2 - 6x + 8x - 4 &= 0 \\ 6x(2x - 1) + 4(2x - 1) &= 0 \\ (6x + 4)(2x - 1) &= 0 \\ 6x + 4 = 0 \text{ or } 2x - 1 &= 0 \\ 6x = -4 \text{ or } 2x &= 1 \\ x = -4/6 \text{ or } x &= 1/2 \\ x = -2/3 \text{ or } x &= 1/2 \end{aligned}$$

If you were able to write the steps it should not be difficult to understand the next example. If not go through the material or discuss it with your tutor or other learners at the learning centre. The example will enable you to understand the method of factorization better as well as build your confidence in in factorization.

Example 4

Solve $x^2 + 3x - 40 = 0$

Solution

$$\begin{aligned} x^2 + 3x - 40 &= 0 \\ x^2 - 5x + 8x - 40 &= 0 \\ x(x - 5) + 8(x - 5) &= 0 \\ (x + 8)(x - 5) &= 0 \\ x + 8 = 0 \text{ or } x - 5 &= 0 \\ 8 = -8 \text{ or } x &= 5 \end{aligned}$$

Here $3x$ is expressed as $-5x + 8x$. -5 and 8 are factors of $1 \times 40 = 40$

We have so far, briefly revised the factorization of quadratic expressions. You have learnt what a quadratic equation is and how to solve quadratic equations by factorization.

Let us now look at another method of solving quadratic equations which is known as completing the square. The knowledge you have acquired on factorization will be very helpful in understanding this method. In fact you have to know how to factorize if you have to successfully solve a quadratic equation using the method we are going to discuss now.

Completing the Square

In the previous section of this unit you learned how to solve quadratic equations by factorization. All quadratic equations that can be solved by factorization can be solved by completing the square but not all the equations that can be solved by completing the square can be solved by factorization. So you can see the advantage of knowing how to solve quadratic equations by completing the square. In fact any quadratic equation with real root can be solved using the method of completing the square. A real root is one which is a real number.

To begin our explanation of this method we first look at a concept that you came across in your study of the junior secondary school course. You learnt that numbers such as 4, 9, 16, 25 and many more are called perfect squares. You realize that:

$$\begin{aligned}4 &= 2 \times 2 \\9 &= 3 \times 3 \\16 &= 4 \times 4 \\25 &= 5 \times 5\end{aligned}$$



Note it!

Note that the numbers 4, 9, 16 and 25 can be obtained by multiplying a whole number with itself.

Obviously not all numbers are perfect squares. Can we obtain a perfect square by adding another number to a non-perfect square? Yes we can. For instance 15 is not a perfect square but we can add 1 to 15 to obtain a perfect square 16. We will now use this idea to explain how we can form perfect squares from quadratic expressions that are not. We will use an example to explain. If you find it difficult to understand the following example, you are encouraged to study the material on factorization again. The purpose of the following example is to illustrate a perfect square.

Example 5

Let us factorize $x^2 - 6x + 9$.

Solution:

$$\begin{aligned}x^2 + 6x + 9 &= x^2 + 3x + 3x + 9 \\&= x(x + 3) + 3(x + 3) \\&= (x + 3)(x + 3) \\&= (x + 3)^2\end{aligned}$$

i.e. $x^2 + 6x + 9 = (x + 3)^2$

$$6x = 3x + 3x$$

$(x + 3)(x + 3) = (x + 3)^2$ just as x multiplied by x is x^2

The expression in example 5 is an example of a perfect square. Here is another perfect square:

$$\begin{aligned}x^2 - 8x + 16 &= x^2 - 4x - 4x + 16 \\ &= x(x - 4) - 4(x - 4) \\ &= (x - 4)(x - 4) \\ &= (x - 4)^2\end{aligned}$$

The above expressions are not the only perfect squares-but there are many more. The method of completing the square involves, first forming a quadratic equation which is a perfect square, then working out the root or roots of the equation.

Now that we have looked at the examples of perfect squares let us now state the general rule to follow in order to come up with a perfect square from a given quadratic equation.

In general the quantity to be added is the square of half of b. For example to form a perfect square from $x^2 + 6x$ we add to it the square of b . In this expression b is equal to 6 and half of 6 is 3. The square of 3 is 9. So we must add 9 to $x^2 + 6x$ to form a perfect square. Thus $x^2 + 6x + 9$ is a perfect square as it was shown in the previous example.

Do the following exercise to see if you can form a perfect square from a given quadratic equation.



Activity 6

Consider the expression $x^2 + 2x$.

$x^2 + 2x$ is not a perfect square. What must be added to make it a perfect square? Write your answer in the space below.

Did you manage to form a perfect square? You can now go ahead and check the feedback.



Feedback

In the case of $x^2 + 2x$, $b = 2$ and half of 2 is 1. The square of 1 is 1. Therefore to make $x^2 + 2x$ a perfect square 1 must be added.

Thus $x^2 + 2x + 1$ is a perfect square since:

$$\begin{aligned} x^2 + 2x + 1 &= x^2 + x + x + 1 \\ &= x(x + 1) + 1(x + 1) \\ &= (x + 1)(x + 1) \\ &= (x + 1)^2 \end{aligned}$$

$x^2 + x = x(x + 1)$ and $x + 1 = 1(x + 1)$ since 1 is always a factor.

We now consider an example that illustrates how we can solve an equation by completing the square.

Example 6

Solve the equation $x^2 + 8x - 33 = 0$

Solution:

$x^2 + 8x - 33 = 0$ can be written as $x^2 + 8x = 33$

Since $b = 8$ half of b is 4. So if we add $4^2 = 16$ to the left hand side of the equation the result will be a perfect square $x^2 + 8x + 16$. To balance the equation we must add 16 to both sides of the equation. Thus, we will obtain:

$$\begin{aligned} x^2 + 8x + 4^2 &= 33 + 4^2 \\ x^2 + 8x + 16 &= 33 + 16 \\ x^2 + 8x + 16 &= 49 \end{aligned}$$

At this stage factorize $x^2 + 8x + 16$

$8x = 4x + 4x$

$$\begin{aligned} x^2 + 8x + 16 &= 49 \\ x^2 + 4x + 4x + 16 &= 49 \\ x(x + 4) + 4(x + 4) &= 49 \\ (x + 4)(x + 4) &= 49 \\ (x + 4)^2 &= 49 \end{aligned}$$

$(x + 4)^2 = (x + 4)(x + 4)$

We get the square root both sides

$$\begin{aligned} \sqrt{(x + 4)^2} &= \sqrt{49} \\ x + 4 &= \pm 7 \\ x + 4 &= 7 \text{ or } x - 4 = 7 \\ x &= 7 - 4 \text{ or } x = -7 - 4 \\ x &= 3 \text{ or } x = -11 \end{aligned}$$

The square root of 49 is +7 or -7. We, therefore have two equations as shown in the next step.

The roots or solutions of the equation in the example are integers -11 and 3. An integer belongs to the set $\{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$. It is not always the case that you will get integers. Here is another example to illustrate this. The example will also strengthen your understanding of the problems we have solving.

Example 7

Solve the equation $x^2 - 4x - 2 = 0$

Solution:

$$x^2 - 4x - 2 = 0$$

$$x^2 - 4x = 2$$

The left hand side of this equation can be made into a perfect square. The half of the coefficient of x is 2. Therefore 2^2 must be added. To balance the equation 2^2 must be added to both sides of the equation.

$$x^2 - 4x + 2^2 = 2 + 2^2$$

$$x^2 - 4x + 4 = 2 + 4$$

$$(x - 2)^2 = 6$$

$$\sqrt{(x - 2)^2} = \sqrt{6}$$

$$x - 2 = 2.449 \text{ or } -2.449$$

$$x = 2.449 + 2 \text{ or } x = -2.449 + 2$$

$$x = 4.449 \text{ or } x = -0.449$$

This number has been rounded off to 3 decimal places. The explanation of how to round off to a required number of decimal places was given in unit 1.

The solutions are decimal numbers. So you see it is not always that you have solutions that are integers.

We look at yet another example. In all the examples looked at so far we have been adding the square of a whole number. In the following example we will add the square of a fraction.

Example 8

Solve the equation $x^2 - 3x + 2 = 0$

$$x^2 - 3x + 2 = 0$$

$$x^2 - 3x = -2$$

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 = -2 + \left(\frac{3}{2}\right)^2$$

$$x^2 - 3x + \frac{9}{4} = -2 + \frac{9}{4}$$

$$x^2 - \frac{3}{2}x - \frac{3}{2}x + \frac{9}{4} = \frac{1}{4}$$

The square of $\frac{3}{2}$, which is half of 3, is added to both sides of the equation

$$\left(\frac{3}{2}\right)^2 = \frac{3}{2} \times \frac{3}{2} = \frac{3 \times 3}{2 \times 2} = \frac{9}{4}$$

$$-3x = -\frac{3}{2}x - \frac{3}{2}x$$

After factorising we get:

$$x\left(x - \frac{3}{2}\right) - \frac{3}{2}\left(x - \frac{3}{2}\right) = \frac{1}{4}$$

Factorising further we get:

$$\left(x - \frac{3}{2}\right)\left(x - \frac{3}{2}\right) = \frac{1}{4}$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{1}{4}$$

We get the square root both sides

$$\left(x - \frac{3}{2}\right) = \sqrt{\frac{1}{4}}$$

$$\left(x - \frac{3}{2}\right) = \pm\frac{1}{2}$$

$$x = \frac{1}{2} + \frac{3}{2} \text{ or } x = -\frac{1}{2} + \frac{3}{2}$$

$$x = \frac{1+3}{2} \text{ or } x = \frac{-1+3}{2}$$

$$x = \frac{4}{2} \text{ or } x = \frac{2}{2}$$

$$x = 2 \text{ or } x = 1$$

The square root of $\frac{1}{4}$ is $+\frac{1}{2}$ or $-\frac{1}{2}$. We, therefore have two equations as shown.

In this section of topic 1 we have discussed how to solve quadratic equations by completing the square. You have seen that any quadratic equation with real roots can be solved by this method. Another method which can be used to solve any quadratic equation with real roots is quadratic formula method. Let us discuss it.

Quadratic Formula

You have already learned about factorization and completing the square methods of solving quadratic equations. In this sub-section of the topic you will learn about the third method which involves applying the quadratic formula. As with completing the square not all quadratic equations that can be solved by formula can be solved by factorization but all equations that can be solve by factorization can be solved by formula. So you can see that this method has the same advantages as those for completing the square.

At the beginning of this unit a quadratic equation was defined as “An equation of the form $ax^2 + bx + c = 0$, where **a**, **b** and **c** are constants and $a \neq 0$, is called a **quadratic equation** in one variable x.” Now taking the general expression of a quadratic equation $ax^2 + bx + c = 0$, the roots can be worked out by using the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let us now look at an example to see at how the quadratic formula can be used to solve a quadratic equation.

Example 9

Solve the equation

$$2x^2 - 5x + 2 = 0$$

$$a = 2, b = -5 \text{ and } c = 2$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(2)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{25 - 4(4)}}{2(2)}$$

The square of a negative number is a positive number. So $-5 \times -5 = 25$

$$x = \frac{5 \pm \sqrt{25 - 16}}{4}$$

$$x = \frac{5 \pm \sqrt{9}}{4}$$

$$x = \frac{5 \pm 3}{4}$$

$$x = \frac{5+3}{4} \text{ or } x = \frac{5-3}{4}$$

5 ± 3 means that 3 is positive or negative.

Thus we have $\frac{5+3}{4}$ and $\frac{5-3}{4}$

$$x = \frac{8}{4} \text{ or } x = \frac{2}{4}$$

$$x = 2 \text{ or } x = \frac{1}{2}$$

Now that you have looked at this example do the following activity to test your comprehension.



Activity 7

Solve the equation $x^2 - 7x + 12 = 0$

Write your answer in the space below.



Feedback

In the equation $x^2 - 7x + 12 = 0$, $a = 1$, $b = -7$ and $c = 12$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

substituting a , b and c with their values we get:

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(12)}}{2(1)}$$

$$x = \frac{7 \pm \sqrt{49 - 4(12)}}{2(1)}$$

$$x = \frac{7 \pm \sqrt{49 - 48}}{2}$$

$$x = \frac{7 \pm \sqrt{1}}{2}$$

$$x = \frac{7 \pm 1}{2}$$

The square root of 1 is +1 or -1.

This results in:

$$x = \frac{7+1}{2} \text{ or } x = \frac{7-1}{2}$$

$$x = \frac{8}{2} \text{ or } x = \frac{6}{2}$$

$$x = 4 \text{ or } x = 3$$

Using this method must be getting very interesting for you. Remember to discuss any difficulties with your tutor at the learning centre or with other learners.

In the previous example and activity the value of **b** which is the coefficient of x is negative. Let us consider another example. In the following example the value of the constant **c** is negative.

Example 10

Solve the equation $12x^2 + 2x - 4 = 0$

Solution

$a = 12$, $b = 2$ and $c = -4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(12)(-4)}}{2(12)}$$

$$x = \frac{-2 \pm \sqrt{4 - 4(-48)}}{2(12)}$$

$$x = \frac{-2 \pm \sqrt{4+192}}{24}$$

$$x = \frac{-2 \pm \sqrt{196}}{24}$$

$$x = \frac{-2 \pm 14}{24}$$

$$x = \frac{-2+14}{24} \text{ or } x = \frac{-2-14}{24}$$

$$x = \frac{12}{24} \text{ or } x = \frac{-16}{24}$$

$$x = \frac{1}{2} \text{ or } x = \frac{-2}{3}$$

This example has been given to show that if a quadratic equation has a solution or solutions it can be solved using this method regardless of the sign (negative or positive) of **a**, **b** and **c**.

So far you have learned about three methods of solving quadratic equations. The three methods are factorization, completing the square and quadratic formula. Now that you have learned about the three algebraic methods of solving quadratic equations in this topic, let us now summarize what we have learned in this first topic of the unit.

Topic 1 Summary



Summary

In this first topic on algebraic methods you have learned about three methods of solving quadratic equations: factorisation, completing the square and quadratic formula methods.

You first learnt about the factorisation method. You learnt that in the factorisation method you first need to work out the factors then work out the values of the variables in the factors. You learnt that for the equation $ax^2 + bx + c = 0$ you have to find the factors of **ac** whose sum is **b**, then express **bx** as the sum of terms using the factors. Thereafter you worked out the values of the variables in the factors.

You also learnt about the method of completing the square. You learnt that for the method of completing the square you have to form a perfect square from the equation you are solving then work out the values of the variable. After learning this you learnt about the quadratic method.

In the quadratic formula method you learnt to use the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for the equation $ax^2 + bx + c = 0$.

You can now go ahead and complete topic exercise 1 in the assignment section of this unit. You are encouraged to work out the solutions to the

questions given before you check the feedback in order to arrive at a correct assessment of your progress.

Let us now turn our attention to the graphical method of solving quadratic equation as we move to the second topic.

Topic 2 Graphical Method

Solving equations by drawing graphs is not completely new to you. In unit 3 on graphs of polynomials you learned how to draw graphs which included graphs of quadratic equations. This method of solving quadratic equations involves drawing the graph of the quadratic equation. The knowledge that you gained in unit 3 will be very helpful in understanding how to get the solutions of quadratic functions.

This topic addresses the second unit outcome which is provided below.

After you have studied this topic you will be able to:



Outcome

Solve quadratic equations graphically.

Solving Quadratic Equations Graphically

In topic 2, of the previous unit, you learnt how to determine the points where a graph passes. You learnt that given an equation and selected values of x you can work out the corresponding values of y by replacing x with a particular value in the equation. For instance when given the equation $y = x^2 + 3x + 2$ we can work out the value of y when $x = 2$. This is done by replacing x with 2 in the equation. Thus:

$$\begin{aligned} y &= x^2 + 3x + 2 \\ y &= 2^2 + 3(2) + 2 \\ y &= 4 + 6 + 2 \\ y &= 12 \end{aligned}$$

Therefore when $x = 2$, $y = 12$. This means that the graph of this equation passes through $(2, 12)$.

For us to draw the graph of any given equation we need to have enough points that will enable us to have a correct shape of the graph. So we should select different values of x and work out the corresponding values of y . Let us look at an example to illustrate this. In the example we will select some values of x and work out their corresponding values of y .

Example 1

Use integer values of x from -3 to 2 to determine the points through which the graph of the equation $x^2 + x - 2 = 0$ passes.

Solution

In giving the solution we will use a table and work step by step. In Table 1 below the first row has selected values of x.

x	-3	-2	-1	0	1	2
x^2						
+x						
-2						
y						

These are from the equation $x^2 + x - 2 = 0$ written one column

There are other values of x and only the ones shown have been picked because these are the numbers specified in the question.

Table 1(a): Integer values of x in $x^2 + x - 2 = 0$

We mentioned earlier that we will work step by step. So we will fill the next row and leave the last three for now as they are. We will now get different values of x^2 for the different values of x.

The numbers in the second row are a result of squaring the numbers in the first. For instance in table 2, $(-3)^2 = 9$ and $(-2)^2 = 4$. Please verify the other four.

x	-3	-2	-1	0	1	2
x^2	9	4	1	0	1	4
+x						
-2						
y						

Table 1(b): Values of x^2 in $x^2 + x - 2 = 0$

We now get different values of +x for the different values of x in the first row. For instance the first number in the first row is -3. So $+(-3) = -3$. Thus we write -3 as the first number in the third column.

x	-3	-2	-1	0	1	2
x^2	9	4	1	0	1	4
+x	-3	-2	-1	0	1	2

-2						
y						

Table 1(c): Values of +x in $x^2 + x - 2 = 0$

The number -2 is constant for all values of x.

x	-3	-2	-1	0	1	2
x^2	9	4	1	0	1	4
+x	-3	-2	-1	0	1	2
-2	-2	-2	-2	-2	-2	-2
y						

-2 is constant for all values of x. As explained above a constant is a number or quantity that does not vary. That is why we have the same number here for all values of x.

Table 1(d): The constant in $x^2 + x - 2 = 0$

We now work out the different values of y by adding the three numbers above in one column. The first value of y is 4 as you can see in table 4. It obtained by working out $9 + -3 + -2$ which gives 4. Similarly the second number 0 is obtained by working out $4 + -2 + -2 = 0$.

x	-3	-2	-1	0	1	2
x^2	9	4	1	0	1	4
+x	-3	-2	-1	0	1	2
-2	-2	-2	-2	-2	-2	-2
y	4	0	-2	-2	0	4

This is the result of adding numbers in the same column in the three rows above. For instance 4 at the end is the result of adding 4, 2 and -2.

Table 1(e): The value of $x^2 + x - 2$

As you can see from the table each x has a corresponding value of y and the two give the coordinates of the point where the graph passes. As you can see from the table when $x = -3$ $y = 4$. This means that the graph passes through $(-3, 4)$. The graph also passes through $(-2, 0)$, $(-1, -2)$, $(0, -2)$, $(1, 0)$ and $(2, 4)$

We have just discussed how to determine the points where a graph passes. We will now use the same equation $x^2 + x - 2 = 0$ in the following example to illustrate how to solve a quadratic equation by drawing its graph.

As an example let us solve the equation $x^2 + x - 2 = 0$. This is the same equation that is in the example above.

Example 2

Let us solve the equation $x^2 + x - 2 = 0$

Solution

The solutions of $x^2 + x - 2 = 0$ are the value of x which results in the equation $x^2 + x - 2$ being zero when x is replaced by that value. For us to know that value of x we come up with a table with selected values of x .

x	-3	-2	-1	0	1	2
x^2	9	4	1	0	1	4
$+x$	-3	-2	-1	0	1	2
-2	-2	-2	-2	-2	-2	-2
y	4	0	-2	-2	0	4

Table 2: Table of values for $x^2 + x - 2 = 0$

For each value of x there is a corresponding value of y . For instance when x is equal to -3 y is equal to 4 . This means that the graph for $x^2 + x - 2 = 0$ passes through the point $(-3, 4)$. In a similar way we can identify the other points where the graph passes.

The other points where the graph passes are $(-2, 0)$, $(-1, -2)$, $(0, -2)$, $(1, 0)$ and $(2, 4)$. The diagram shows the points where the graph passes.

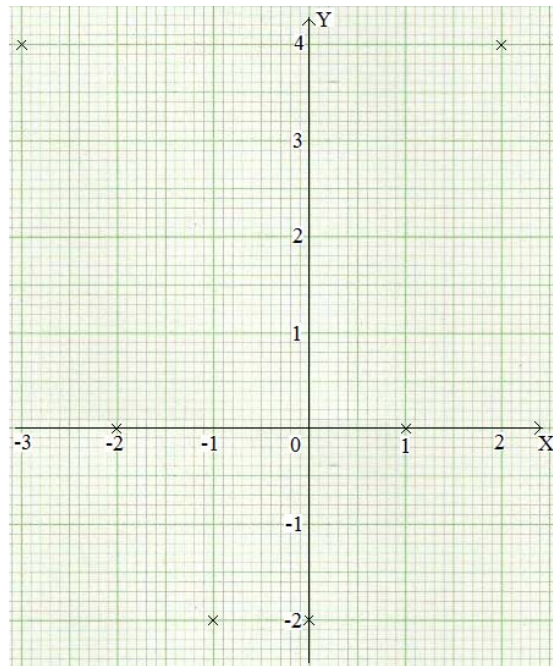


Figure 1(a): Locating the points $(-2, 0)$, $(-1, -2)$, $(0, -2)$, $(1, 0)$ and $(2, 4)$

We can now use the points indicated to draw the graph of the equation. Join the points by drawing a smooth curve. The graph will appear as follows.

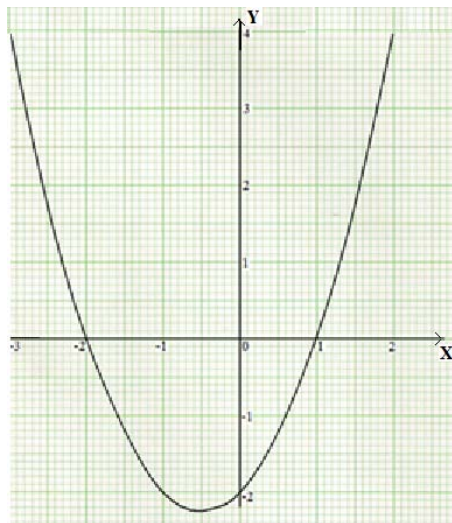
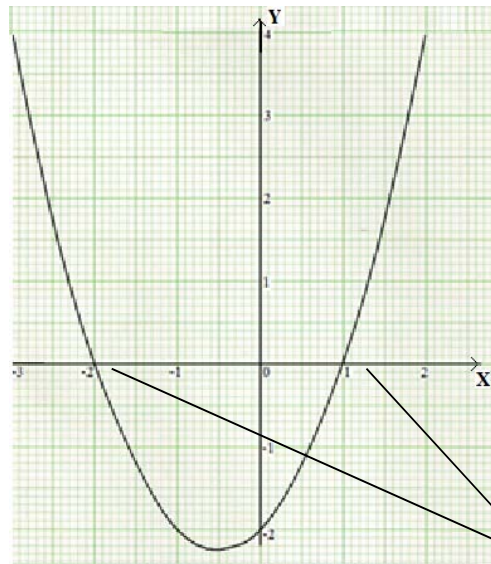


Figure 1(b): Joining the points $(-2, 0)$, $(-1, -2)$, $(0, -2)$, $(1, 0)$ and $(2, 4)$

The solutions of the equation are the points where the curve meets the x-axis.



The solution of the equation.

Figure 1(c): Identifying the solution of $x^2 + x - 2 = 0$

The solutions are:

$$x = -2 \text{ or } x = 1$$

You have seen how you can determine the coordinates of the points through which a curve passes and how to locate these points. Let us look at another example to strengthen your understanding.

Example 2

Let us solve the equation

$$x^2 + 5x + 6 = 0$$

Solution

We come up with a table and use selected values of x. We take integers from -5 to 1. The table will be as follows.

x	-5	-4	-3	-2	-1	0	1
x^2	25	16	9	4	1	0	1
5x	-25	-20	-15	-10	-5	0	5
+6	+6	+6	+6	+6	+6	+6	+6
y	6	2	0	0	2	6	12

Table 3: Table of values for $x^2 + 5x + 6 = 0$

The graph of the quadratic equation will be as follows:

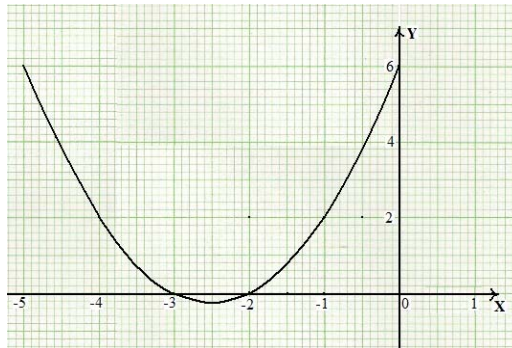


Figure 2: The graph of $x^2 + 5x + 6 = 0$

The solutions lie at the points where the graph crosses the x-axis. Therefore the solutions are $x = -3$ or $x = -2$.

After studying and understanding this example and the one before, practice your skill of solving quadratic equations using graphical method by doing the following activity.



Activity 1

Solve the equation $x^2 - 5x + 6 = 0$. You will find that it is not difficult if you follow the procedure explained in the examples. Write your answer in the space that follows.



Feedback

Each of these is a product of 5 and the number in the first row in the same column.

We make a table with selected values of x and work out the corresponding values of y .

x	0	1	2	3	4	5
x^2	0	1	4	9	16	25
$-5x$	0	-5	-10	-15	-20	-25
$+6$	+6	+6	+6	+6	+6	+6
y	6	2	0	0	2	6

Each of these is a square of the number in the first row in the same column.

Each of these is the sum of the three numbers above in the same column.

Table 4: Table of values for $x^2 - 5x + 6 = 0$

The graph passes through $(0, 6)$, $(1, 2)$, $(2, 0)$, $(3, 0)$, $(4, 2)$, and $(5, 6)$. These points are shown in figure 5.

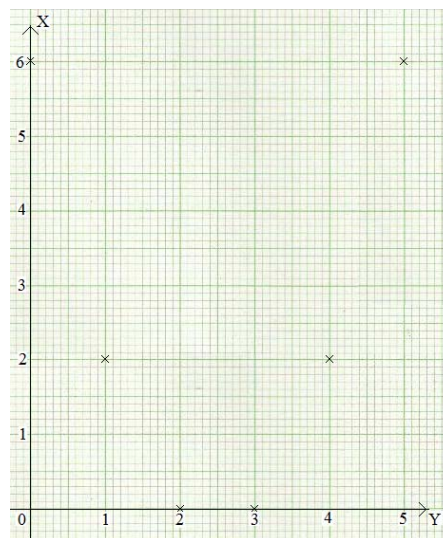


Figure 3: Locating the points $(0, 6)$, $(1, 2)$, $(2, 0)$, $(3, 0)$, $(4, 2)$, and $(5, 6)$

We now draw the graph passing through these points.

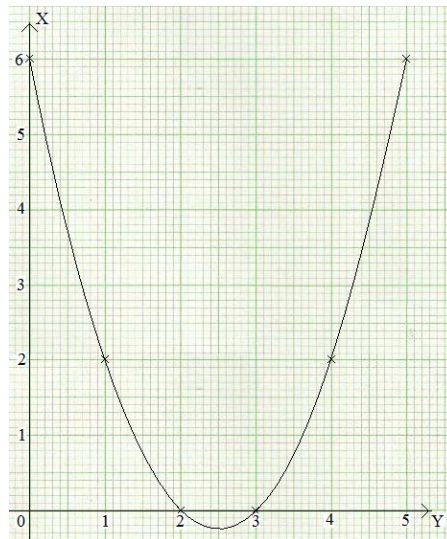


Figure 4: The graph of $x^2 - 5x + 6 = 0$

The solutions of the equation are: $x = 2$ or $x = 3$.

Was this activity easy? If you found it difficult revise the examples and then discuss your challenges with your tutor and other learners at the study centre.

We have been considering the graphical method of solving quadratic equations. We will now give a summary of what we have looked at in this topic.

Topic 2 Summary



Summary

Topic 2 of this unit has covered the graphical method. You learnt how to draw graphs of quadratic equations. You learnt that when you are given the equation $ax^2 + bx + c = 0$ you can work out the corresponding value of y for each value of x and locate the point on the xy plane using the same values of x and y . We discussed that for us to have an accurate graph we need to have enough points where the graph passes. For this reason we select enough values of x and work out the corresponding value of y for each x .

When the graph is drawn the points where the graph crosses the x -axis are noted and those are taken to be the solutions of the equation.

The methods you have learnt about in the first and second topics will prove to be helpful to you because quadratic equations are applied in a number of situations as you will see in the next topic. But before you proceed to the next topic find out how well you have understood topic 2 by doing the topic 2 exercise in the assignment section. Remember to mark your work by checking it against the feedback provided. If you are

happy with your progress, continue to topic 3. If not, revise the topic and discuss your difficulties with other learners and your tutor.

Topic 3 Application of Quadratic Equations

We have so far looked at four ways of solving quadratic equation. You have learned about the factorization, completing the square, quadratic formula and graphical methods. In this last topic of unit 4 you will learn how you can apply quadratic equations to solve problems. Once you learn this you will be in a better position to solve a variety of problems on real life situations. You will be able to see how certain situations can be interpreted and presented in mathematical statements. For instance you can apply quadratic equations to work the number of tiles needed for a particular room.

So far we have discussed in detail the methods of solving quadratic equations. We will now concentrate on discussing how we can apply quadratic equations. You will see just how interesting this is.

This topic we will address the last unit outcome. After studying this unit you will be able to:



Outcomes

- *Apply* quadratic equations to solve mathematical problems.

Application of Quadratic Equations

As already mentioned earlier, the methods of solving quadratic equations have been discussed in detail in the first two units. We will now concentrate on the application of quadratic equations.

At times you may come across mathematical statements that lead to quadratic equations. Consider the following example.

Example 1

Find two numbers which differ by 3 and have a product of 54.

Solution

The two numbers differ by 3. This means that one is bigger by 3. So we let the two numbers be x and y . Also let x be less than y . Then if we subtract x from y the result is 3. In short $y - x = 3$.

From the equation $y - x = 3$ we get

$$y = 3 + x$$

We know that the product of the two numbers is 54 therefore

$$x \times y = 54$$

Now $x \times y = 54$ is the same as $x \times (3 + x) = 54$. This is because $y = 3 + x$.

Simplification of the left hand side gives:

$$3x + x^2 = 54$$

Subtracting 54 from both sides results in:

$$x^2 + 3x - 54 = 0$$

This quadratic equation is now solved by factorisation which was explained in topic 1.

$$x^2 + 9x - 6x - 54 = 0$$

$$x(x + 9) - 6(x + 9) = 0$$

$$(x - 6)(x + 9) = 0$$

$$x - 6 = 0 \text{ or } x + 9 = 0$$

$$x = 6 \text{ or } x = -9$$

$x = -9$ is invalid so we only take $x = 6$

From above $y = 3 + x$

$$\text{So } y = 3 + x = 3 + 6 = 9$$

Therefore the two numbers are 6 and 9.

You can see how quadratic equations can help you when solving problems such as the one in example 1. Let us look at a second example. It is similar to the first in that it is about two unknown measurements whose difference and product have been given.

Example 2

The length of a classroom is 2 metres longer than the width. Its area is 48m^2 . Find the length and width of the room.

Solution

We let the width of the room be x . Since the length is 2 metres longer, the length is $x + 2$.

The area of the classroom is obtained by multiplying the width and the length.

Thus Area = Length \times Width

Now since Length \times Width = Width \times Length we have

$$x \times (x + 2) = 48$$

$$x(x + 2) = 48$$

$$x^2 + 2x = 48$$

$$x^2 + 2x - 48 = 0$$

$$x^2 - 6x + 8x - 48 = 0$$

$$x(x - 6) + 8(x - 6) = 0$$

$$(x + 8)(x - 6) = 0$$

$$x + 8 = 0 \text{ or } x - 6 = 0$$

$$x = -8 \text{ or } x = 6$$

This is because the area is 48m^2 .

We take $x = 6$ since length has to be positive. Therefore the width is 6 metres and the length is $x + 2 = 6 + 2 = 8$ metres.

In the above two examples the difference and product of two unknown numbers or quantities were given. In the next one what have been given are the sum and the product two unknown numbers.

Example 3

Two numbers have the sum of 16 and the difference of their squares is 32. Find the numbers.

Solution

Since we do not know the numbers we take the numbers to be x and y .

We have been told that the sum of the two numbers is 16 so when we add them we will get 16. Thus:

$$x + y = 16$$

$$y = 16 - x$$

The square of y is y^2 and the square of x is x^2 . We have been told that the difference of their squares is 32. So

$$y^2 - x^2 = 32$$

$$(16 - x)^2 - x^2 = 32$$

$$(16 - x)(16 - x) - x^2 = 32$$

$$16(16 - x) - x(16 - x) - x^2 = 32$$

From above $y = 16 - x$. So we substitute y with this value of y

$$256 - 16x - 16x + x^2 - x^2 = 32$$

$$256 - 32x = 32$$

$$32x = 256 - 32$$

$$32x = 224$$

$$x = 7$$

$$y = 16 - x$$

$$y = 16 - 7$$

$$y = 9$$

From above

Substitute the x
with the value of x
above which is 7

Therefore the numbers are 7 and 9.

The examples that we have discussed should now give you enough confidence to do the following activity.



Activity

A certain man started his own business when he was forty years old and his first born son was seventeen years old. Some years have past since the man started his business. If the product of the ages of the man and his son right now is 860, how old is the man and his son?

Were you indeed as confident about doing this activity as we suggested above? Ongoing practice of the skills being taught in this unit will eventually lead to confidence and deeper understanding of how to solve quadratic equations. Read our feedback below to check your answer.



Feedback

We do not know how many years have past since the man started the business. So let us take the number of years that have past to be x . Then the man is $40 + x$ years old and the son is $17 + x$ years old. We know that the product of their ages is 860. Therefore the product of $40 + x$ and $17 + x$ is 860. As an equation this is expressed as:

$$(40 + x) \times (17 + x) = 860 \text{ or as:}$$

$$(40 + x)(17 + x) = 860$$

$$40(17 + x) + x(17 + x) = 860$$

$$680 + 40x + 17x + x^2 = 860$$

$$680 + 40x + 17x + x^2 - 860 = 860 - 860$$

$$680 + 40x + 17x + x^2 - 860 = 0$$

$$x^2 + 40x + 17x + 680 - 860 = 0$$

$$x^2 + 57x - 180 = 0$$

$$x^2 - 3x + 60x - 180 = 0$$

$$x(x - 3) + 60(x - 3) = 0$$

$$(x - 3)(x + 60) = 0$$

$$(x - 3) = 0 \text{ or } x + 60 = 0$$

$$x = 3 \text{ or } x = -60$$

Multiply the terms in the first brackets with those in the second.

Remember x is taken as the number of years that have past. So we exclude -60 conclude that 3 years have past since the man started his own business.

Therefore the man is 43 years old and the son is 20 years old.

There are many other mathematical problems that can lead to quadratic equations. You should now be able to apply the knowledge and skills you have acquired in this topic which we now give a summary of below.

Topic 3 Summary



Summary

In this topic you have learnt how to apply quadratic equations. You learnt that from real life situations we can come up with mathematical expressions and equations. In the process of solving these equations sometimes quadratic equations are formed. These equations are then solved using methods which you learnt about. These are factorisation, completing the square, quadratic formula and graphical methods.

The method of factorisation involves first working out the factors of quadratic equations and then working out the values of the variables in the factors that will result in the factor being zero. You also learnt about the method of completing the square. You learnt that for the method of completing the square you have to form a perfect square from the equation you are solving. For the quadratic formula method you learnt how to use the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for the equation $ax^2 + bx + c = 0$.

You were able to see that one has more than one alternative on the choice of the method to use at any particular time. These are the methods which you learnt about in the previous two topics.

You are reminded to do topic exercise 3 in the assignment section. Once you have completed the exercise you can go ahead and check the feedback for the exercise in the feedback section.

The summary below gives you a broad picture of what we studied in this unit. You can read it now or after completing the topic exercise. Once you have completed these two tasks: reading the topic summary and answering and marking the topic exercise; you will then be ready to study the next unit. Unit 5 is ratio, proportion and rate.

Unit 4 Summary



Summary

The fourth unit on quadratic equations had three topics. The first topic discussed solving quadratic equations algebraically. The second one covered solving quadratic equations graphically while the third dealt with the application of quadratic equations.

In the first topic you learned about three methods of solving quadratic equations. You first learned about the factorisation method which involves factorising the quadratic equation as the name suggests. You then learned about using the method of completing the square. This method requires that one forms a perfect square from a given quadratic equation. This method can be used to solve any quadratic equation which has real roots. The third method that you learned about is the use of the quadratic formula. Like the method of completing the square this method can be used to solve any quadratic equation which has got real roots.

The second topic addressed the graphical method of solving quadratic equations. When this method is used the graph of the equation is drawn and the solution is taken to be the point at which the graph meets the x-axis.

The third topic discussed the application of quadratic equations. In this topic you were able to see how mathematical problems which lead to quadratic equations can be solved are applied in real life situations.

Unlike several other units in this course this unit has no tutor marked assignment.

Have you already completed your topic 3 exercise? Good. If you have not yet done so, do that now.

Here is a list of references for further reading.

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Assignment



Assignment

This part consists of three topic exercises. Each exercise is based on the topic discussed in this unit. You are encouraged to do the exercises before you look at the feedback provided. This will enable you to correctly assess your understanding of each topic.

Topic 1 Exercise



Exercise

1. Solve the following equations by factorization.

- (a) $x^2 + 3x + 2 = 0$
- (b) $2x^2 + x - 3 = 0$
- (c) $3x^2 + 10x + 8 = 0$
- (d) $3x^2 - 5x + 2 = 0$
- (e) $4x^2 + x - 5 = 0$

2. Solve the following equations by completing the square:

- (a) $x^2 - 5x + 6 = 0$
- (b) $x^2 - 8x + 15 = 0$
- (c) $x^2 - 4x + 3 = 0$

3. Solve the following equation by quadratic formula.

- (a) $18x^2 + 3x - 15 = 0$
- (b) $2x^2 + 3x - 2 = 0$
- (c) $x^2 + 8x - 33 = 0$

Topic 2 Exercise



Exercise

Solve the following equations by drawing their graphs:

- 1. $x^2 - 7x + 12 = 0$
- 2. $-x^2 + 8x - 15 = 0$
- 3. $-x^2 - x + 6 = 0$

Topic 3 Exercise



Exercise

- The difference in the floor area of two square rooms is 20m^2 . One room is 2m longer each way than the other. Find the dimensions of the rooms.
- A boy is 3 years older than his sister and the product of their years is 154. Find their ages.
- A rectangular piece of cardboard measures 14cm by 16cm . Strips of equal width are glued to the cardboard. The cardboard has a new area of 323cm^2 . Find the width of the strips glued to the cardboard.
- The length of a rectangle is 3 times longer than the width. Its area is 147cm^2 . Find the dimensions of the rectangle.
- The base and the height of a triangle are equal and the area is 18cm^2 . Find the dimensions of the triangle.

Feedback

Topic 1 Exercise Feedback



Feedback

- $$x^2 + 3x + 2 = 0$$

$$x^2 + x + 2x + 2 = 0$$

$$x(x + 1) + 2(x + 1) = 0$$

$$(x + 2)(x + 1) = 0$$

$$x + 2 = 0 \text{ or } x + 1 = 0$$

$$x = -2 \text{ or } x = -1$$
 - $$2x^2 + x - 3 = 0$$

$$2x^2 - 2x + 3x - 3 = 0$$

$$2x(x - 1) + 3(x - 1)$$

$$(2x + 3)(x - 1) = 0$$

$$2x + 3 = 0 \text{ or } x - 1 = 0$$

$$2x = -3 \text{ or } x = 1$$

$$x = -3/2 \text{ or } x = 1$$
 - $$3x^2 + 10x + 8 = 0$$

$$3x^2 + 4x + 6x + 8 = 0$$

$$x(3x + 4) + 2(3x + 4) = 0$$

$$(x + 2)(3x + 4) = 0$$

$$x + 2 = 0 \text{ or } 3x + 4 = 0$$

$$x = -2 \text{ or } x = 4/3$$
 - $$3x^2 - 5x + 2 = 0$$

$$3x^2 - 3x - 2x + 2 = 0$$

$$\begin{aligned}
 3x(x-1) - 2(x-1) &= 0 \\
 (3x-2)(x-1) &= 0 \\
 3x-2=0 \text{ or } x-1 &= 0 \\
 3x=2 \text{ or } x &= 1 \\
 x = \frac{2}{3} \text{ or } x &= 1
 \end{aligned}$$

$$(e) 4x^2 + x - 5 = 0$$

$$4x^2 - 4x + 5x - 5 = 0$$

$$4x(x-1) + 5(x-1) = 0$$

$$(4x+5)(x-1) = 0$$

$$4x+5=0 \text{ or } x-1=0$$

$$4x = -5 \text{ or } x = 1$$

$$x = -\frac{5}{4} \text{ or } x = 1$$

$$2. (a) x^2 - 5x + 6 = 0$$

$$x^2 - 5x = -6$$

$$x^2 - 5x + \left(\frac{5}{2}\right)^2 = -6 + \left(\frac{5}{2}\right)^2$$

$$x^2 - 5x + \frac{25}{4} = -6 + \frac{25}{4}$$

$$x^2 - \frac{5}{2}x - \frac{5}{2}x + \frac{25}{4} = \frac{1}{4}$$

$$x\left(x - \frac{5}{2}\right) - \frac{5}{2}\left(x - \frac{5}{2}\right) = \frac{1}{4}$$

$$\left(x - \frac{5}{2}\right)\left(x - \frac{5}{2}\right) = \frac{1}{4}$$

$$\left(x - \frac{5}{2}\right)^2 = \frac{1}{4}$$

$$\left(x - \frac{5}{2}\right) = \sqrt{\frac{1}{4}}$$

$$\left(x - \frac{5}{2}\right) = \pm \frac{1}{2}$$

$$x = \frac{1}{2} + \frac{5}{2} \text{ or } x = -\frac{1}{2} + \frac{5}{2}$$

$$x = \frac{1+5}{2} \text{ or } x = \frac{-1+5}{2}$$

$$x = \frac{6}{2} \text{ or } x = \frac{4}{2}$$

$$x = 3 \text{ or } x = 2$$

$$(b) x^2 - 8x + 15 = 0$$

$$x^2 - 8x = -15$$

$$x^2 - 8x + 4^2 = -15 + 4^2$$

$$x^2 - 8x + 16 = -15 + 16$$

$$x^2 - 4x - 4x + 16 = 1$$

$$x(x - 4) - 4(x - 4) = 1$$

$$(x - 4)(x - 4) = 1$$

$$(x - 4)^2 = 1$$

$$(x - 4) = \sqrt{1}$$

$$(x - 4) = \pm 1$$

$$x = 1 + 4 \text{ or } x = -1 + 4$$

$$x = 5 \text{ or } x = 3$$

$$(c) x^2 - 4x + 3 = 0$$

$$x^2 - 4x = -3$$

$$x^2 - 4x + 2^2 = -3 + 2^2$$

$$x^2 - 4x + 4 = -3 + 4$$

$$x^2 - 4x + 4 = 1$$

$$\sqrt{(x - 2)^2} = \sqrt{1}$$

$$x - 2 = \pm 1$$

$$x - 2 = 1 \text{ or } x - 2 = -1$$

$$x = 1 + 2 \text{ or } x = -1 + 2$$

$$x = 3 \text{ or } x = 1$$

$$3. (a) 18x^2 + 3x - 15 = 0$$

$$a = 18, b = 3 \text{ and } c = -15$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(18)(-15)}}{2(18)}$$

$$x = \frac{-3 \pm \sqrt{9 - 4(-270)}}{2(18)}$$

$$x = \frac{-3 \pm \sqrt{9+1080}}{36}$$

$$x = \frac{-3 \pm \sqrt{1089}}{36}$$

$$x = \frac{-3 \pm 33}{36}$$

$$x = \frac{-3+33}{36} \text{ Or } x = \frac{-3-33}{36}$$

$$x = \frac{30}{36} \text{ Or } x = \frac{-36}{36}$$

$$x = \frac{5}{6} \text{ Or } x = -1$$

$$(b) 2x^2 + 3x - 2 = 0$$

$$a = 2, b = 3 \text{ and } c = -2$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-2)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{9 - (-4)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{9+16}}{4}$$

$$x = \frac{-3 \pm \sqrt{25}}{4}$$

$$x = \frac{-3 \pm 5}{4}$$

$$x = \frac{-3+5}{4} \text{ Or } x = \frac{-3-5}{4}$$

$$x = \frac{2}{4} \text{ Or } x = \frac{-8}{4}$$

$$x = \frac{1}{2} \text{ Or } x = -2$$

$$(c) x^2 + 8x - 33 = 0$$

$$a = 1, b = 8 \text{ and } c = -33$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-8 \pm \sqrt{64 - 4(1)(-33)}}{2(1)}$$

$$x = \frac{-8 \pm \sqrt{64 - 4(-33)}}{2(1)}$$

$$x = \frac{-8 \pm \sqrt{64 + 132}}{2}$$

$$x = \frac{-8 \pm \sqrt{196}}{2}$$

$$x = \frac{-8 \pm 14}{2}$$

$$x = \frac{-8 + 14}{2} \text{ or } x = \frac{-8 - 14}{2}$$

$$x = \frac{6}{2} \text{ or } x = \frac{-22}{2}$$

$$x = 3, \text{ or } x = -11$$

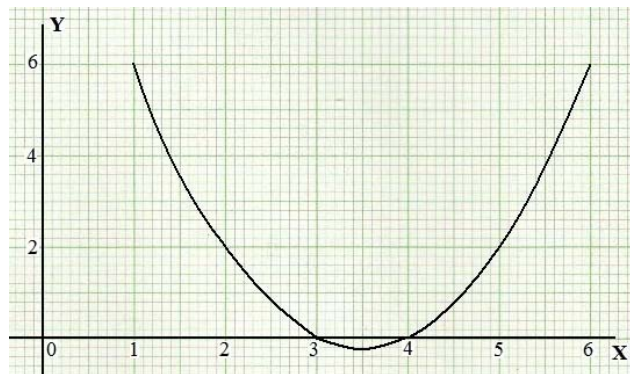
Topic 2 Exercise Feedback



Feedback

1. $x^2 - 7x + 12 = 0$

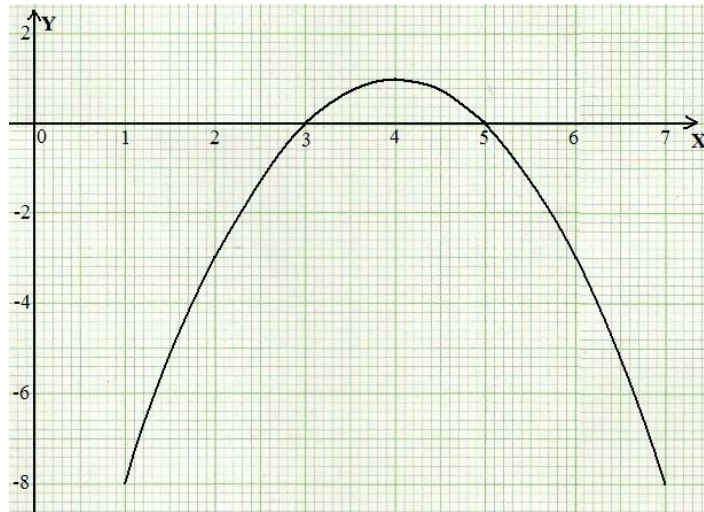
x	1	2	3	4	5	6
x^2	1	4	9	16	25	36
$-7x$	-7	-14	-21	-28	-35	-42
+12	+12	+12	+12	+12	+12	+12
y	6	2	0	0	2	6



The graph meets the x-axis at $x = 3$ and $x = 4$. The solutions are: $x = 3$ or $x = 4$.

2. $-x^2 + 8x - 15 = 0$

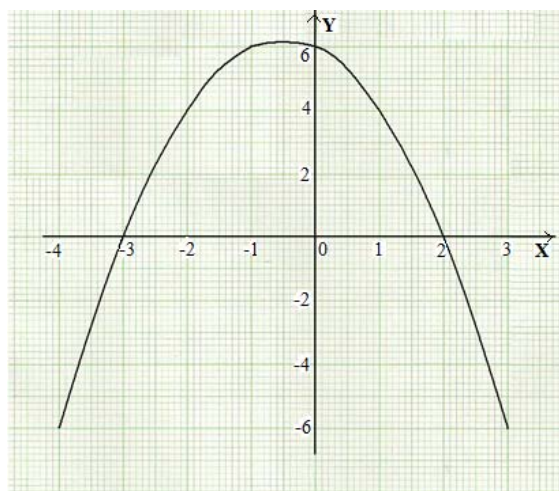
x	1	2	3	4	5	6	7
$-x^2$	-1	-4	-9	-16	-25	-36	-49
+8x	8	16	24	32	40	48	56
-15	-15	-15	-15	-15	-15	-15	-15
y	-8	-3	0	1	0	-3	-8



The solution of the equation is $x = 3$ or $x = 5$ because the curve meets the x-axis at 3 and 5.

3. $-x^2 - x + 6 = 0$

x	-4	-3	-2	-1	0	1	2	3
$-x^2$	-16	-9	-4	-1	0	-1	-4	-9
$-x$	4	3	2	1	0	-1	-2	-3
+6	+6	+6	+6	+6	+6	+6	+6	+6
y	-6	0	4	6	6	4	0	-6



The solution of the equation is $3 = -3$ or $x = 2$ since the graph meets the x-axis at the points where the value of x are -3 and 2.

Topic 3 Exercise Feedback



Feedback

1. Let the length of the floor with a shorter length be x and the length of the longer floor be $x + 2$.

The area of the smaller room is $x \times x = x^2$.

The area of the bigger room is $(x + 2)(x + 2) = x^2 + 4x + 4$.

Subtract the area of the smaller room from the area of the bigger room.

$$x^2 + 4x + 4 - x^2$$

The difference in the area of the two rooms is 20. Therefore

$$x^2 + 4x + 4 - x^2 = 20$$

$$x^2 - x^2 + 4x + 4 = 20$$

$$4x + 4 = 20$$

$$4x = 16$$

$$x = 4$$

Therefore the length of the floor of the smaller room is 4 metres while the length of the floor of the bigger room is $4 + 2 = 6$ metres.

2. The girl is 3 years younger, so if the boy is x years old then the girl is $x - 3$ years old.

The product of their years is 154. So if we multiply x and $(x - 3)$ we get 154. This means

$$x \times (x - 3) = 154$$

$$x^2 - 3x - 154 = 0$$

$$x^2 - 14x + 11 - 154 = 0$$

$$x(x - 14) + 11(x - 14) = 0$$

$$(x + 11)(x - 14) = 0$$

$$x + 11 = 0 \text{ or } x - 14 = 0$$

$$x = -11 \text{ or } x = 14$$

The boy is 14 years old and the girl is $14 - 3 = 11$ years old.

3. We do not know the width of the strip so we let it be x . When strips of equal width are added the width and length of the board increase by x .

The width is now $(14 + x)$ and the length is $(16 + x)$

The new length is obtained by multiplying $14 + x$ by $16 + x$, which we have been told is 323. Thus:

$$(16 + x) \times (14 + x) = 323$$

$$224 + 14x + 16x + x^2 = 323$$

$$x^2 + 30x + 224 - 323 = 0$$

$$x^2 + 30x - 99 = 0$$

$$x^2 - 3x + 33x - 99 = 0$$

$$x(x - 3) + 33(x - 3) = 0$$

$$(x + 33)(x - 3) = 0$$

$$x + 33 = 0 \text{ or } x - 3 = 0$$

$$x = -33 \text{ or } x = 3$$

Therefore the width of the strips glued was 3 cm.

4. Let the width of the rectangle be x . Then the length will be $3x$ since the length is 3 times longer than the width. Since the area of the rectangle is 147cm^2 , $3x \times x = 147$.

$$\text{Thus } 3x^2 = 147$$

$$x^2 = 49$$

$$x = \pm \sqrt{49}$$

$$x = \pm 7$$

$$x = 7 \text{ or } x = -7$$

Length is never negative. Therefore the width is 7 centimetres. We know that the length is three times the width. So the length is 21 centimetres.

5. Let the base and the height be equal to x .

In unit 11 the area of plane figures is discussed in more detail. In that unit you learnt that the area of the triangle is given by:

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$$

Since the area of the triangle is 18cm^2 , and the area and the base are both equal to x the area will be

$$\text{Area} = \frac{1}{2} \times x \times x = 18$$

$$\text{Area} = \frac{1}{2} x^2 = 18$$

$$\frac{1}{2} x^2 = 18$$

$$2 \times \frac{1}{2} x^2 = 2 \times 18$$

$$x^2 = 36$$

$$\sqrt{x^2} = \sqrt{36}$$

$$x = 6$$

The base and the height of the triangle are both equal to 6 centimetres.

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Unit 6

Variation

Introduction

This is unit 6 in your mathematics 11 course. In unit 5, you covered ratios, proportion and rate. You learnt how to solve problems involving ratios, direct and inverse proportions. You went further to learn how to apply the concepts of ratios and proportion to everyday life situation.

In this unit, you will learn about variation. The concepts you learnt in unit 4 on quadratic equations and unit 5 on ratio, proportion and rate will apply here very much. You will not do something completely different from what you learnt in unit 5. It is the same content but on a higher level. In unit 4, you learnt how to solve equations. The skill of solving equations will be very helpful in this unit. Variation is a function that relates the values of one variable to those of another variable. In Variation, we are looking at a relationship that exists between two variables. Remember that in unit 2, you learnt about relations between two sets. You learnt that one variable is related to another variable by a rule. This knowledge is also very helpful in studying this unit. The question is, 'what kind of relation exists between the two given variables?' Your knowledge of ratios, proportion and rate and on relations and mappings will play an important role in variation.

In your study of mathematics 11, all units are very important. There is no unit that is less in importance as compared to the others. This means that you have to pay particular attention to each unit. Allocate the same commitment, time and resources to all the units. Remember that all units are inter related. Knowledge learnt in one unit is applied in another.

There are two topics in this unit. The name of the topic (Variation) may sound different and new but the contents are the same as in unit 5 and were covered in Junior Certificate Program. In each topic, you will be required to do some activities and topic exercises. You are encouraged to answer all the questions in the topic exercises.

At the end of this unit, you will be required to do tutor marked assessment (TMA) 2 which you will send to your tutor for marking. This TMA is the second in the course. You did the first one after unit 3. This one will cover unit 4, unit 5 and this unit.

After studying this unit, the following outcomes should be achieved.

Upon completion of this unit you will be able to:

Unit Outcome



- *Express* a given variation mathematically using symbols.
- *Change* a given variation expressed mathematically into an equation.
- *Distinguish* between Inverse and Direct Variation.
- *Solve* problems on Direct and Inverse Variation.
- *Draw* graphs of Direct and Inverse Variation.
- *Interpret* graphs of Direct and Inverse Variation.
- *Solve* problems based on the other forms of direct and inverse variation of the form $y = x^n$ where $-3 \leq n \leq 3$.
- *Solve* problems based on Joint Variation.
- *Solve* problems on Partial Variation.

Timeframe



It is estimated that to complete studying this unit you will need between 10 to 14 hours. You will take about 6 hrs on the first topic and about 8 hrs on the second topic. The second topic will take longer because of the introduction to joint and partial variations. You have not learnt joint variation and partial variation anywhere else in your study from grade 8 to grade 10. This makes it a bit new and hence, will require more time from you to study them. Do not worry if you take longer in finishing this unit. This is because we all have different abilities and hence we study at different paces.

You are encouraged to spend 2 hours answering each topic exercise in this unit. Since there are two topic exercises in this unit, this means that you will spend 4 hours on these exercises.

You will be required to spend 3 hours on the tutor marked assignment. Ensure that you take exactly the time allocated as this helps you to practice management of time during examinations.

The total hours for completing the unit will thus be between 17 and 21 hours.

Learning Resources



Like in the previous units, you will need the following materials:

- Ruler
- Pencil
- Graph Paper
- Eraser

Teaching and Learning Approaches



The emphasis is on your understanding the concept of variation. This will be achieved by working out problems involving variations through your interpretation of the relationship between variables and the translating into mathematical language or models. You will be able to master the skills as you practice the skills in working out the activities and exercises as an individual alone or in a group with others studying the same unit.

As you study this unit, you will come across spaces left in the unit. These spaces are meant for you to use as you interact with the materials. You need to use them to write down some notes as you study or even work out certain questions in the activities given to you. As you study the topics, it is necessary for you to assess yourself. Assessing your knowledge and skills acquired, you have to work out the activities and the exercises and mark your own work using the feedback provided immediately. It is expected of you not to go through the feedback before doing the activity.

Terminology



Inverse:	The term refers to the reversing of the effect of an operation.
Joint Variation:	This is the type of variation in which the value of one variable depends on the values of two or more variables.
Partial variation:	The word means not complete. Here it means that the variation does not completely depends on one variable.
Proportion:	Two pairs of numbers are said to be in proportion if the ratio formed by the first pair equals the ratio formed by the second pair.
Proportional:	One variable is said to be in proportional to another if the ratio of corresponding values remains constant.
Varies :	The term means to change under the same conditions.
variation	The term means to change in value. The quantities involves change in value under certain conditions.

Topic 1 Direct and Inverse Variation

You will learn how to express a given variation mathematically using symbols. Variation as a term is new. However, the content is not new to you as you learnt about them in the Junior Certificate Program and in unit 5 of mathematics 11 when you dealt with ratios and proportion. This time you will learn this topic further in detail. You will learn to define variation. You will learn how to connect variables by using the variation symbol. You will also learn how to form the equation arising from the situation given and solving it. You will further learn how to distinguish between direct and inverse variation through your interaction with the materials.

After studying this topic, you will be required to answer self assessment questions at the end of the topic. You are encouraged not to go through the feedback before doing the topic exercises.



In this topic, we will address the first six of the nine unit outcomes, which are:

- *Express* a given variation mathematically using symbols.
- *Change* a given variation expressed mathematically into an equation.
- *Distinguish* between inverse and direct variation.
- *Solve* problems on direct and inverse variation.
- *Draw* graphs of direct and inverse variation.
- *Interpret* graphs of direct and inverse variation.

To achieve these outcomes, the following objectives should be achieved by the end of this topic. You should be able to:



- *Write* a given variation using symbols.
- *Write* a given relationship as an equation.
- *Find* the constant of variation.
- *Distinguish* between direct and inverse variation.

Defining Variation

As earlier stated, you learnt about ratios and proportion in unit 5. You learnt how the two quantities relate to each other as ratios. In ratios, you were comparing quantities of the same kind or unit. In proportion, you learnt that two ratios are equal. You learnt how to use ratios and proportions to solve problems. Now, the questions you need to ask yourself are: What is variation all about? What is involved in variation? We will try to define variation and answer the questions above as we consider the following situation.

Let us consider the following situation given in a table below:

P	3	6	9	12	15	X
Q	1	2	3	4	5	6

Table 1: Showing values of P and Q

We are going to use the concept of ratios. By the way, a concept is defined as an IDEA which is expressed in symbols. If we use the concept of ratios, the two variables can be expressed as $P : Q$ which is also written as $\frac{P}{Q}$. Substituting the values from the table in the expression, we get the following:

$$\frac{3}{1} = 3; \frac{6}{2} = 3; \frac{9}{3} = 3; \frac{12}{4} = 3; \frac{15}{5} = 3.$$

You should notice that all the answers in each situation are the same. All the divisions are giving 3 as an answer. We say that 3 is common throughout. In such a case, 3 is said to be a constant. This also means that when we do the last division, we should still get 3.

From the above, we say that $\frac{x}{6} = 3$. To get the value of x, we cross multiply 6 by 3:

$$\text{Therefore, } x = 6 \times 3 = 18.$$

Since the ratios in each case are the same (3) as shown above, we can conclude that P is proportional to Q. The word proportional has already been defined in the way it is used in mathematics. But other definitions can also help us understand proportional. The word proportional in another way is defined as the relationship between two corresponding quantities. Using these definitions, we can then say that P corresponds to Q or we can say that P relates to Q in some way. Therefore, to say that P is proportional to Q means that P corresponds to Q or P is related to Q under certain conditions. This relationship is now shown in the following statements:

The relationship between P and Q is expressed as $P = 3Q$ (P is 3 times Q) which is also expressed as $\frac{P}{Q} = 3$ (by dividing by Q both sides).

In general, the statement $\frac{P}{Q}$ is called the ratio. The ratio $\frac{P}{Q}$ is a constant represented by the letter, k. By constant, we mean the same number appearing throughout the answers. It is not changing in value from one value to another. The same number is appearing in every ratio. In mathematics, this constant can be represented by any letter. Except that in variation, we use the letter, k.

The k is called the constant of variation.

In the above example, 3 is a numerical value and P and Q are letters called variables; we need to represent 3 by a letter that will stand for any value that can be calculated for. In that case as stated above, we use the letter, k. Therefore, $\frac{P}{Q} = k$ or $P = kQ$.

Now, be it known to you that you should always find the value of the constant of variation, k and replace it in the final equation as shown in $P = 3Q$, where 3 is the value of k.

From the table 1 above, you should notice that as P increases Q increases as well. In a situation where one variable increases as the corresponding variable increases, we say that there is a relationship between the variables. This relationship is expressed as: $P \propto Q$, where \propto is the symbol of variation. The statement $P \propto Q$ is read as, 'P varies directly as Q.'

The term **varies** leads us to the word **variation**, which is a noun and it means **to change** in value. The quantities (variables) involved **change in value** at given conditions. Using our example, we say that P changes in value as Q changes in value. The changing can either be both increasing in values or both decreasing in values. This is what is called **direct variation**.

Direct Variation

From the statement in the definition of variation, we have introduced variation in general. We have said P varies directly as Q. This has been understood as meaning P corresponds directly to Q. We have shown in the example above the connection of variables in direct variation. You are now going to learn how this connection comes up in direct variation as we consider the following situation.

If we know that the cost of 5 exercise books is K 25 000, then we can be able to find the cost of any number of books. For example, 10 books of the same type will cost K 50 000, 1 book will cost K 5 000, 2 books will be K 10 000 and so on. We can see that if the number of books increases, the cost of the number of books will also increase. If the number of books reduces the cost also reduces. We can show this in the following table below.

Number of books (N)	Cost in Kwacha(C)
1	5 000
2	10 000
3	15 000
4	20 000
5	25 000
10	50 000
35	175 000
50	250 000

Table 2: Cost of books illustrating direct proportional

In the situation illustrated above, we can say that the number of books is directly proportional to the cost of books. The more books we buy, the higher the total cost of books and the fewer the number of books we buy, the lesser the cost. The term proportional was very much used in the junior classes, in Grade 9 and in unit 5 of mathematics 11. We will use the terms proportional and variation interchangeably. By this we mean we will use ‘direct proportional’ to mean ‘varies directly’ or use ‘varies directly’ to mean ‘direct proportional’. From the table above, we can conclude and say that the number of books varies directly to the cost.

The following example will help you calculate the missing value of a variable given using variation.

As stated earlier, the example is to help you relate what has been said in the situation and explain the process of finding the constant of variation and enable you to work on the future problems with confidence.

Example 1

y varies directly as x . When $y = 6$, $x = 3$. Find the value of the constant of variation k and calculate y when $x = 12$.

Solution

In the question, the situation is in word form and it has to be translated into the mathematical format by using the appropriate symbols.

In this example, you have to first translate situation outlined above, into the language of variation as: $y \propto x$.

Then write it in the form of the equation introducing the constant of variation k connecting y and x as: $y = kx$.

Now find the value of the constant of variation k by substituting y with 6 and x with 3.

$$6 = k \times 3$$

$$6 = 3k$$

$$\frac{6}{3} = \frac{\cancel{3}}{\cancel{3}}k$$

$$2 = k$$

Now we take 2 and we replace k with it in the equation and we have

$$y = 2x$$

This will now help us answer the question of finding y when $x = 12$.

$$\text{We have } y = 2 \times 12$$

$$y = 24.$$

In this example, we were introduced to the process of finding the constant of variation and then using it to form the equation connecting the two variables. We then used the equation to find the missing value of the other variable given.

Divide both sides by 3 to find the value of k

The next example is to consolidate the knowledge and skills acquired in finding the constant of variation and solving for the missing value of the variable given.

In the example, the mathematical language is already used. The connection of the two variables, p and q , are already done. All we need to do is to calculate for the constant of variation and then use the equation to work out the question to find the missing values of the variables. This will help you appreciate the process and be able to use it in any situation.

Example 2

In a certain experiment $p \propto q$ so that

$P = kq$. It is known that $p = 40$ when $q = 16$.

Find the constant of variation k and hence calculate:

- (1) p when $q = 24$
- (2) q when $p = 32$

Solution

In this example, the mathematical language of variation is already used in $p \propto q$ and $p = kq$. All we need to do is to find the value of k by using the equation $p = kq$.

First let us find the value of the constant of variation k by replacing p with 40 and q with 16.

$$40 = 16k$$

$$\frac{40}{16} = \frac{\cancel{16}}{\cancel{16}}k$$

$$\frac{5}{2} = k$$

$$k = \frac{5}{2} \text{ or } 2\frac{1}{2}$$

Divide both sides by 16 to find the value of k .

Reduce the fraction $\frac{40}{16}$ to its lowest term ($\frac{5}{2}$) by dividing into each value by a highest common factor 8.

Our equation now is $P = \frac{5}{2}q$.

We can now use this equation to answer the two parts of the question.

1. p when $q = 24$

$$p = \frac{5}{2} \times 24$$

$$p = 60$$

2. q when $p = 32$

$$32 = \frac{5}{2}q$$

$$32 = \frac{5q}{2}$$

Cross multiply: 2×32

$$32 \times 2 = 5q$$

$$64 = 5q$$

Divide both sides by 5

$$\frac{64}{5} = \frac{5q}{5}$$

$$12\frac{4}{5} = q$$

$$\text{Hence } q = 12\frac{4}{5} \text{ or } q = 12.8$$

We have done example two and have seen how to calculate for the missing values of the variables. The next activities are designed to assist you practice the skills acquired so far.



Now that we have done 2 examples, you are asked to tackle the following activity. This activity is to help you to practise the working skills that you have learnt from the two examples.

Activity 1

Given that y varies as the square of x and $y = 8$ when $x = 4$, find y when $x = 5$.

Work out this on a separate sheet of paper.



Here is the feedback for the work above. You should check it through after you have done the activity.

Feedback

In the activity, the situation is in what is referred to as word problem. Your duty is to translate into mathematical language shown above in the examples so that it becomes easy to work with mathematically.

You should have first translated into the language of variation by using symbol \propto . The statement should have been; $y \propto x^2$.

Then write the equation: $y = kx^2$.

So you can see that whether the constant of variation k is mentioned in the question or not it should be there in all statements mathematically.

Substitute y with 8 and x with 4 to find the value of k .

We have $8 = k \times 4^2$

$$\frac{8}{16} = \frac{\cancel{16}}{\cancel{16}} k$$

$$\frac{1}{2} = k$$

So the equation is $y = \frac{1}{2}x^2$. Use it to find the value of y when $x = 5$.

$$y = \frac{1}{2} \times 5^2$$

$$y = \frac{1}{2} \times 25$$

$$y = 12\frac{1}{2} \text{ or } y = 12.5.$$



Another activity for more practice. This will also help you in understanding the concept. Work out this question on a separate sheet of paper. Then compare your answers with those provided in the feedback below.

Activity 2

Given that A varies directly as the square root of B and $A = 4$ when $B = 4$. Find the value of the constant of variation k and find A when $B = 9$.



The following is the feedback for the activity above. Only compare your answers to it after having done the activity.

Feedback

In this activity, the situation is given in word problem. Your duty here again is to translate into mathematical language as you did in the above activity so that it becomes easy to work with mathematically.

You should first have translated into language of variation $A \propto \sqrt{B}$.

$$A = k\sqrt{B}$$

$$4 = k\sqrt{4}$$

$$4 = k \cdot 2$$

$$\frac{4}{2} = \frac{2}{2}k$$

$$2 = k.$$

So the value of the constant of variation is 2. This now helps you make the equation. Since we know that $k = 2$, we substitute in the equation $A = k\sqrt{B}$ and we get $A = 2\sqrt{B}$

Remember never forget to substitute values given. Here you substitute A with 4 and B with 4

To find A when $B = 9$, you should have used the equation.

$A = 2\sqrt{B}$ and substitute B with 9:

$$A = 2 \times \sqrt{9}$$

$$A = 2 \times 3$$

$$A = 6.$$

We have done the activities. Now, here is an example on how to draw the graph of the relation between variables.

The next example is on being able to draw the relationship of direct variation. This example requires knowledge on international currencies and usage. By international, we mean currencies used in other countries either locally (within a country) or can be used in other nations (between Zambia and Zimbabwe, for example). The word currency is a term used to refer to the generally accepted medium of exchange (for example, dollar-USA, Rand-SA, Kwacha-Zambia) used in a particular nation. The example requires connection of the relation between the two variables. This will help you appreciate the process and be able to use it in any situation.

Example 3

The value, V , of a piece of meat is proportional to the square of its weight, W . If a piece of meat weighing 10 grams is worth £200, find:

- (a) The value of a piece of meat weighing 30 grams
- (b) The weight of a piece of meat worth £5000

Solution

Just like in the other examples, you need to follow all the other steps learnt so far. In this case, you still have to see the relationship as a variation and form an equation. Here is the procedure:

The relationship is: $V \propto W^2$

Then you form the equation as: $V = kW^2$ where k is a constant of variation

You have to find the value of k by substituting the following: $V = 200$, $W = 10$ in the equation $V = kW^2$ to find the value of k

$$V = kW^2$$

$$200 = k(10^2)$$

$$200 = k \times 100$$

$$\frac{200}{100} = \frac{100k}{100}$$

$$2 = k$$

Since now we have the value of k as 2, we replace k with this value of 2 in the equation. Therefore, the equation is now: $V = 2W^2$

(a) When $W = 30$, $V = 2 \times (30^2) = 2 \times 900 = 1\,800$

(b) When $V = 5\,000$,

$$V = 2W^2$$

$$5000 = 2W^2$$

$$\frac{5000}{2} = \frac{2W^2}{2}$$

$$2500 = W^2$$

$$\sqrt{2500} = \sqrt{W^2}$$

$$50 = W$$

This presentation above can also be shown in form of a graph which would show the relationship between V and W .

Now, that we have answered the example, we need to show how the relationship of V and W can be shown on a graph. In unit 3 on graphs of polynomials, you learnt how to draw a graph. You learnt how to draw a linear graph, a quadratic graph and further on a cubic graph. The skills learnt of drawing graphs are very important here as you begin to deal with drawing of graphs. You will need that skill in drawing these graphs here.

Relationship can also be explained in graphs and it makes the work easy to do this. When we explain relations using numbers without graphs, some situations are difficult to understand. The use of a graph makes it easier to understand. We have to draw the graph of this relationship. To draw the graph of this relationship, you have to have a table of values. Now considering what is been discussed, you can never have negative values. The values to consider will be from 0 only because the items under discussion are never in negatives in the real life. Since V is running in thousands, it will take values from 1000 to 5000. W is running in tens, it will take values from 10 to 50. This means that the scale will be as follows: 2 cm to 1000 units

on the Vertical axis and 2 cm to 10 units on the horizontal axis.

The values W are to be substituted in the equation to find the values of V. The table will have the following nature after the substitution:

W	10	20	30	40	50
V	200	800	1800	3200	5000

Table 3: Showing values of V and W

The graph of the above relationship will be as follows:

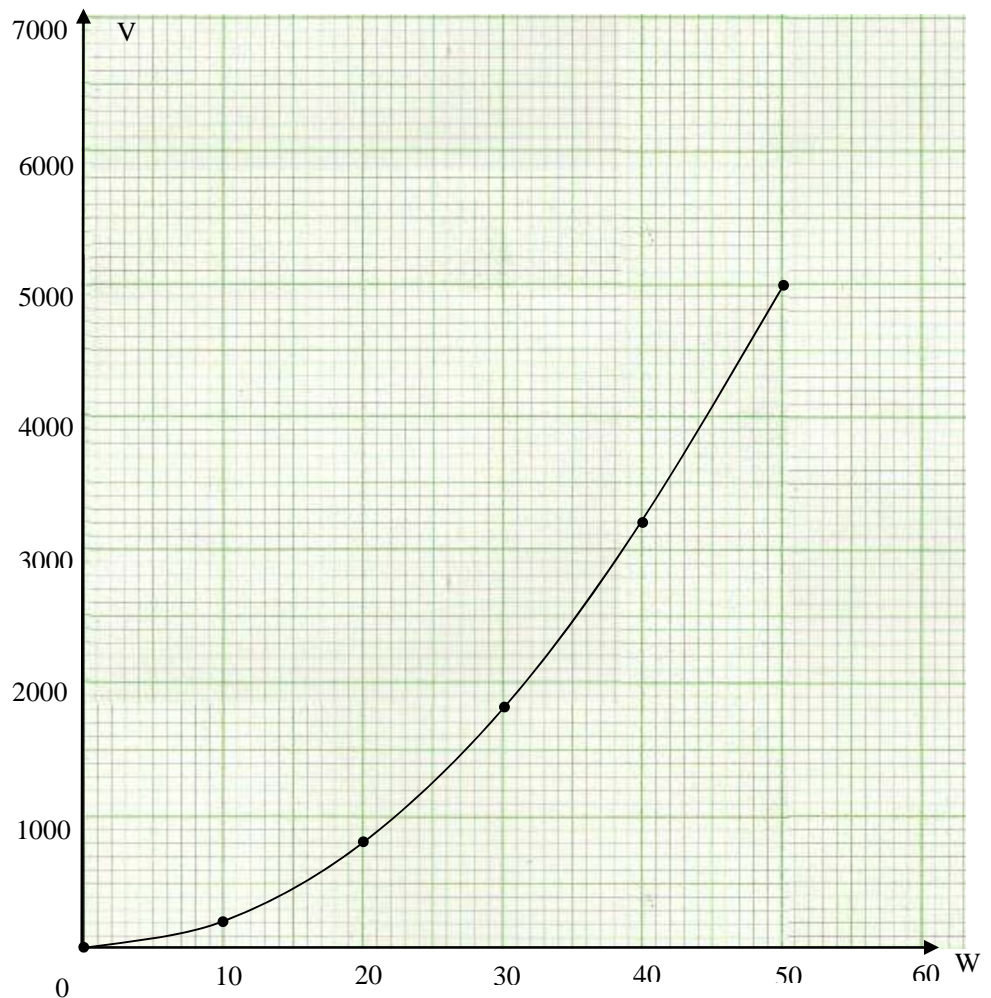


Figure 1: Showing relationship between V and W

The graph here shows the relationship of the value of a piece of meat to its weight. It is a direct variation in the sense that both are

increasing. As the value of weight increase, the value V (cost) increases.

This brings us to the end on the section on direct variation. The situations discussed so far were to explain direct variation of variables. In this first section of this topic, you were introduced to the symbol for variation and the direct variation. You also learnt that in direct variation if one side increases the other side also increases in the same ratio and vice versa.

You also learnt that it is important to find the value of the constant 'k' first. You went further to learn how to solve problems involving direct variation.

Now we are going to explore another aspect of variation which is inverse variation.

Inverse Variation

In *the* section of this topic, we shall learn about inverse variation.

In the terminology section, we defined inverse variation. However, we can also say that **inverse** simply means **reverse**. By reverse, it is meant that one variable increases in value while the other variable reduces in value. Take note of the conditions under which the increase or reduction is occurring. It is not just any increase or any reduction. The following illustrations explain this point.

Consider the following situation. If 10 Pupils take 4 days to slash grass on a certain portion of the school grounds, how many pupils should be taken to slash the same portion in 2 days?

You can clearly see here that the number of days has been reduced. So we obviously need more pupils to slash the same portion in those 2 days. We can see that when one quantity is increased the other quantity is reduced. When the increase and decrease is in the inverse (reciprocal) ratio, we say that there is an **INVERSE RELATIONSHIP** between the quantities.

The term **reciprocal** is another word used to mean **inverse**. In mathematics, the reciprocal (inverse) of 2 is $\frac{1}{2}$. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$. The reciprocal of $\frac{1}{3}$ is 3. In other words, a reciprocal (inverse) is a number which is a multiplicative inverse of another number. The two numbers when multiplied together gives the product of 1.

Now, to explain this concept of inverse variation further, consider the following situation:

A packet contains 128 sweets to be shared between two children. Each child will get 64 sweets. If we double the number of children

to 4, each child gets 32 sweets. Again if we double the number of children from 4 to 8, each child gets 16 sweets. Note that the total number of sweets does not change as we increase the number of children. What changes is the number of sweets each child gets. This number reduces as the number of children increase as shown in the table 2 below:

Number of Children	Number of sweets per child	Total number of sweets
2	64	128
4	32	128
8	16	128
16	8	128
32	4	128
64	2	128

Table 4: Showing the inverse relationship

If we pick the first two entries on both sides and form two ratios as shown below:

$$\begin{array}{cc} \text{Number of children} & \text{Number of sweets per child} \\ \frac{2}{4} & \frac{64}{32} \end{array}$$

When reduced to their lowest terms, we get:

$$\frac{1}{2} \qquad \frac{2}{1}$$

The two final ratios are reciprocals of each other. As earlier stated, the term reciprocal simply means the multiplicative inverse. This means that when the two are multiplied together, the answer is 1.

In such a situation, we shall say that the number of children varies inversely to the number of sweets each child gets. In inverse variation, one side increases while the other side reduces in the reciprocal ratio.

If two quantities p and q vary in such a way that when one increases the other decreases in the reciprocal ratio or the other way round, then the two are said to show inverse variation.

When p varies inversely as q , this can be written as $p \propto \frac{1}{q}$. As an

equation we write $p = \frac{k}{q}$ or $k = pq$.

Now, look at the following example. This example will help you understand the procedure of calculating for the constant of variation, k , under inverse variation.

Example 4

y varies inversely as x and $y = 6$ when $x = 4$. Find the value of the constant of variation and find y when $x = 12$.

Solution

First translate into the language of variation as $y \propto \frac{1}{x}$

Then we introduce the constant of variation, k and write an equation as follows: $y = \frac{k}{x}$

(Replace y with 6 and x with 4)

$$6 = \frac{k}{4}$$

$$k = 24$$

The equation becomes:

$$y = \frac{24}{x}$$

Substitute 24 into k in the equation, you get this:

This is the equation connecting y and x and can be used to find y when $x = 12$.

$$y = \frac{24}{x}$$

$$y = \frac{24}{12}$$

$$y = 2.$$

Now, you do another example. This also is to help you master the technique of solving such problems.

Example 5

Given that y varies inversely as x and $y = 8$ when $x = 3$. Find:

- (i) the equation connecting y and x
- (ii) y when $x = 84$
- (iii) x when $y = 9$

Solution

First translate into the language of variation $y \propto \frac{1}{x}$

Then write as an equation

$$y = \frac{k}{x}$$

$$8 = \frac{k}{3}$$

Replace y with 8 and x with 3

$$k = 24.$$

The equation becomes: $y = \frac{24}{x}$

This is the equation connecting y and x .

$$(i) \quad y = \frac{24}{84}$$

To find the value of y when $x = 84$, you substitute x with 84 in the equation as shown here:

$$y = \frac{2}{7}$$

You then divide into both numbers by the highest common factor for both numbers which is 12 and get this:

$$(ii) \quad 9 = \frac{24}{x}$$

$$9x = 24$$

$$x = \frac{24}{9}$$

$$x = \frac{8}{3} \quad \text{or} \quad x = 2\frac{2}{3}$$

In this one, you substitute y with 9 and cross multiply by x and 9;

You divide both sides by 9 and then again divide into both numbers by the highest common factor of 24 and 9 which is 3 as shown in the working:



Here are 2 activities for you to work on. The activities are exactly as the examples given above. They are both dealing with inverse variation situations. You will need to use the knowledge you have learnt so far in solving these problems. They are meant for you to practice the skill of solving these types of problems. Write your answers on a separate sheet of paper.

Activity 3

Given that c varies inversely as the Square of d and $c = 48$ when $d = 3$, calculate c when $d = 4$.



When you have finished the activity, then you can compare your answers to those provided below in the feedback.

Feedback

From the activities above, you have gained knowledge and acquired skills to work this activity out. You should have seen that the relationship of the variables involved here is an inverse relationship. As in the other activities above you have to translate the language into the mathematical one to make your work easy.

First translate into the language of variation

$$c \propto \frac{1}{d^2} \text{ then write as an equation } c = \frac{k}{d^2}$$

$$48 = \frac{k}{3^2}$$

$$k = 48 \times 9$$

$$k = 432.$$

Replace 48 into c and 3 into d . This is done when we are finding the value of k

Replacing k with the value 432 into the equation, we form a complete equation that can be used.

$$\text{Therefore, the equation becomes: } c = \frac{432}{d^2}$$

Use this equation to find the value of c when $d = 4$. You substitute the value of d in the equation:

$$c = \frac{432}{4^2}$$

$$c = \frac{432}{16}$$

$$c = 27.$$



The following activity is similar or same as the other one above. The only difference is in the use of the letters. Remember that different letters can be used at any other time. You should not get confused by such use of different letters. It will help you to consolidate your skills in solving this type of problems. Write your answers on a separate sheet of paper.

Activity 4

P is inversely proportional to \sqrt{q} . If $p = 5$

$q = 16$, find:

- (i) p when $q = 100$
- (ii) q when $p = 60$



When you have finished working out the activity, compare your answers with those provide in the feedback below

Feedback

You have dealt so far with activities on inverse variation. You should practice these skills in this activity. You might have seen

that the relationship of the variables involved here is an inverse relationship. As in other activities above you have to translate the language into the mathematical one to make your work easy.

First translate into the language of variation.

$$p \propto \frac{k}{\sqrt{q}}$$

The equation is $p = \frac{k}{\sqrt{q}}$

$$5 = \frac{k}{\sqrt{16}}$$

$$5 \times \sqrt{16} = k$$

$$5 \times 4 = k$$

$$k = 20$$

$$p = \frac{20}{\sqrt{q}} \text{ equation connecting } p \text{ and } q.$$

$$(i) \quad p = \frac{20}{\sqrt{100}}$$

$$p = \frac{20}{10}$$

$$p = 2.$$

$$(ii) \quad 60 = \frac{20}{\sqrt{q}}$$

$$\frac{60\sqrt{q}}{60} = \frac{20}{60}$$

$$\sqrt{q} = \frac{1}{3}$$

$$(\sqrt{q})^2 = \left(\frac{1}{3}\right)^2$$

$$q = \frac{1}{9}$$

Now, you do another example. This also is to help you master the technique of solving such problems. It is also to help you see how these problem solving skills in inverse variation can be used to solve any problem in our everyday situations.

Example 6

When the temperature is constant, the pressure of a given mass of a gas, P varies inversely as the volume, V .

- (a) Given that the pressure is 75 cm of mercury when the volume is 150 cm^3 , find the pressure when the volume is 100 cm^3 .
- (b) Show this relationship in the graph.

Solution

As usual you have to first determine the relationship between the variables.

In your physical science 11, unit 1, you learnt about Boyle's Law of Kinetic Theory of Matter. You learnt that when the pressure of a gas in a system is increased, the volume is reduced. If the pressure is (reduced), the volume is increased. These conditions stated here, occur under constant temperature. This is the knowledge you need to apply here to relate the two variables.

The relationship is: $P \propto \frac{1}{V}$. The equation now become: $P = \frac{k}{V}$

Using the values $P = 75$ and $V = 150$, you can find the value of the constant variation, k , as follows:

$$75 = \frac{k}{150}$$

$$75 \times 150 = k$$

$$11\,250 = k.$$

Now that you have the value of k as 11250, we use it to complete the equation as: $P = \frac{11250}{V}$. This equation is now very useful to us to find the values of interest at any time as requested and shown in the work below.

We have been asked to find the value of the pressure when the volume is 100 cm^3 .

To find the pressure when volume is 100, you substitute $V = 100$ into the equation:

$$P = \frac{11\,250}{100} = 112.5$$

Therefore, the pressure is 112.5 cm when the volume is 100 cm^3

The above information can be illustrated in a graph of the variables P and V in this situation. Having found the value of P when V is 100 cm^3 , we can now use these values to draw the graph of this relationship under constant (unchanging) temperature.

To draw the graph you need the table of values as shown below. These values were calculated by substituting the values of V in the equation above. Again here it should be noted that there is no room for negative values. We do not talk about negative volume or pressure. These negative values do not exist here. Hence the values of volume will be from 0 to 50 and those of pressure will be from 0 to 1500. The scale is 2 cm to 10 units on the horizontal axis and 2 cm to 500 units on the vertical axis.

V	10	20	30	40	50
P	1125	562.5	375	281.25	225

Table 5: Showing values of Pressure and Volume

The graph of the relationship will be as shown below:

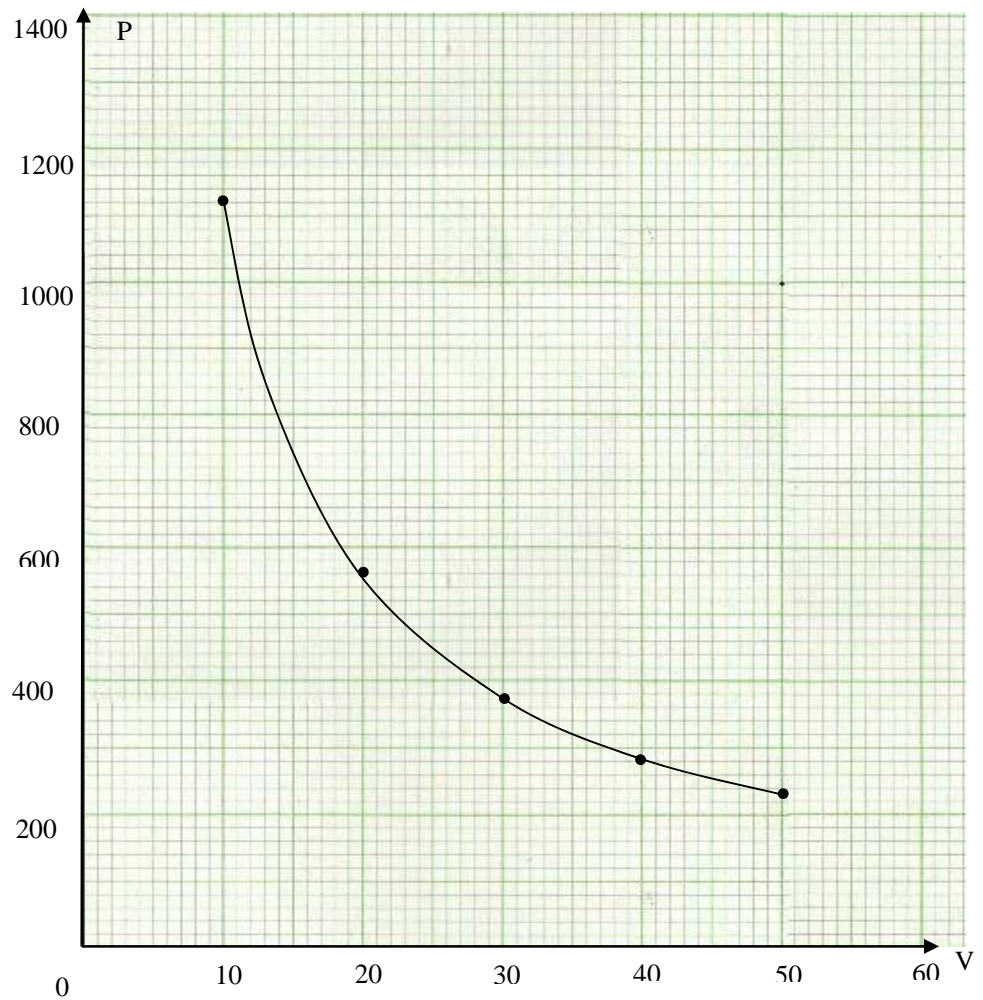


Figure 2: Showing the relationship between P and V

The graph here shows the relationship of the volume of a gas in a system to the pressure exerted on the system. It is an inverse variation in the sense that when one quantity increase, the other decreases. From the graph it should be noted that as the volume increases in value the pressure reduces in value.

You have so far learnt about direct variation. We saw that when one quantity increases the other quantity also increases. You also learnt about the inverse variation in which one quantity increases while the other reduces (decreases) in the reciprocal ratio.

You should by now have known the difference between the direct variation and the inverse variation as explained above.

Topic 1 Summary



As you have now come to the end of this topic, we can now review what you have studied so far. Remember that we had planned to cover a number of outcomes. In this section we expressed a given variation mathematically using symbols. This was done when you learnt about the symbol for variation. You were introduced to the symbol for variation \propto . You went on to learn of how to change a given variation into an equation. We have seen that under direct variation, if one part increases the other part will also increase in the same ratio. We have also seen how we can interpret the whole question into the language of variation. You also went on to learn how to calculate the constant of variation.

Afterwards, you learnt about inverse variation. We have seen that in inverse variation one part should increase while the other part decreases in the inverse ratio. We have seen that in inverse

variation, the statement is written as $y \propto \frac{1}{x}$. We have also learnt

that just like in direct variation, the value of the constant of variation k should be found first.

You learnt how to solve problems on direct and inverse variation and how to illustrate the relationship on a graph. The graphs were of different forms such as straight line graphs and of the form $y = x^n$ where $n = 2$ and 3 . You further learnt how to interpret the graphs of direct and inverse variation. Finally, you learnt to distinguish between inverse and direct variation.

In the next topic, we will look at joint variation and partial variation. In the terminology section, we defined joint variation as the combination of the direct and the inverse variation. We also defined partial variation as the variation of one variable with another and a constant. We will first consider the variation that connects direct variation and inverse variation together. We will then go on to learn about partial variation as the last part of topic 2.

Now, you are ready to attempt the first exercise. Do topic 1 exercise in the assignment section before proceeding to the next topic. In this exercise, you are required to answer all the questions without checking with the answers provided at the end.

Topic 2 Joint and Partial Variation

In the previous topic, you learnt about direct and inverse variation. You learnt how to illustrate the relationship between variables in graphs. You went further to learn the difference between the two variations.

In this topic, you will learn how to form equations connecting the variables in partial variations and joint variations. Both these variations are relatively new to you as you might not have learned about them in Junior Certificate program. However, you will be dealing with these variations in detail to help you master this type of variations. You should remember that the concept of variation has already been tackled. Here you will learn how to combine the different variations in one statement. You will also learn how to solve such situational problems. After studying the topic you will be required to answer self assessment questions at the end of the topic. You are encouraged not to go through the feedback before doing the topic exercises.

It is important for you to understand that the concept of joint and partial variations is very useful in our everyday living. We use them in estimating certain values that we need to use. For example, in estimating the cost of holding a conference, it is the concept of partial variations that is used to determine the total costs expected to be incurred by the organisers. We defined partial variation as the variation of a quantity with another quantity and a constant. You should remind yourself on the definition of joint variation from the terminology section.



We will address the last two of the nine unit outcomes, which are:

- *Solve* problems based on Joint Variation.
- *Solve* problems on Partial Variation.

The above outcomes will be achieved through the following objectives which should be achieved at the end of this topic. At the end of the topic you should be able to:



- Draw graphs of Partial Variation.
- Solve problems involving Joint and Partial Variation.

Joint Variation

If y varies jointly as x and z , it means that y varies directly as x and also y varies directly as z . This can be written separately as: $y \propto x$ and $y \propto z$. Then we can join the two together as $y \propto xz$. A variation in which one variable depends on two or more other variables is known as **joint variation**.

Now, look at the following example. This example and the others to follow later on are to help you master the skills of working with joint variation.

Example 1

y varies jointly as x and z . Find y when $x = 1$ and $z = 2$.

Solution

This is the first example under joint variation. We will use it to show the procedure and methods of solving such problems. You might have seen that the relationship of the variables involved here is a joint relationship. It is a joint variation in the sense that the one variable depends on the two other variables. Since they are two, we first separate them as shown here below:

First translate into the language of variation.

$$y \propto x \text{ and } y \propto z$$

Then joining the two as one statement: $y \propto xz$

From this one, we now form an equation involving the three variables as below:

$$\text{The equation is } y = kxz$$

Substituting the values of y , x and z , we find the value of k ;

$$8 = k \times 2 \times 3$$

$$8 = 6k$$

Divide both sides by 6

$$\frac{8}{6} = \frac{\cancel{6}k}{\cancel{6}}$$

$$\frac{4}{3} = k$$

The equation becomes; $y = \frac{4}{3}xz$

To find the value of y when $x = 1$ and $z = 2$, we substitute the values in the equation:

$$y = \frac{4}{3}xz$$

$$y = \frac{4}{3} \times 1 \times 2$$

$$y = \frac{8}{3} \text{ or } y = 2\frac{2}{3}$$

The second example is the combination of a direct variation situation and an inverse variation situation. This, too, is to help you work with such problems.

Example 2

y varies directly as x and inversely as z . When $y = 2$, $x = 4$ and $z = 6$, find the constant of variation. Then find y when $x = 2$ and $z = 9$.

Solution

From the first example above, you have seen how to go about the situation of joint variation. The three variables have to be connected together in an equation.

First translate into the language of variation

$$y \propto x \text{ and } y \propto \frac{1}{z}$$

Joining the two situations together, we get the following: $y \propto \frac{x}{z}$

$$\text{The equation is } y = \frac{kx}{z}$$

To find the value of k , we substitute the values of y , x and z into the equation and get: $2 = \frac{4k}{6}$

$$4k = 12$$

We cross multiply 2 by 6 and get this:

$$k = 3$$

Then divide both sides by 4 to find the value of k

We then substitute k with the value of k which is 3 in the equation and get the following equation:

Then the equation becomes: $y = \frac{3x}{z}$

Now, to find the value of y when $x = 2$ and $z = 9$, we substitute the values mentioned here in the equation as shown below:

$$y = \frac{3 \times 2}{9}$$

$$y = \frac{6}{9}$$

$$y = \frac{2}{3}$$



Now, here is an activity for you to do. This is to help you master the skill of working with such problems.

Work on separate answer sheets and then compare your answers with those provided in the feedback given below for each activity.

Activity 1

Given that y varies directly as the square of x and inversely as the z . Find the constant of variation when $y = 6$, $x = 2$ and $z = 1$. Find y when $x = 3$ and $z = 2$.



You have to compare your work with the work below in the feedback.

Feedback

Having done two examples to explain the technique in joint variation, it is hoped that you have gained knowledge and acquired skills to work this activity out. As in other examples above, you have to translate the language into the mathematical one to make your work easy.

First translate into the language of variation

$$y \propto x^2 \text{ and } y \propto \frac{1}{z}$$

Then joining the two situations together; $y \propto \frac{x^2}{z}$

$$\text{The equation is } y = \frac{x^2 k}{z}$$

To find the value of k , substitute y with 6, x with 2 and z with 1.

$$6 = \frac{2^2 k}{1}$$

$$6 = 4k$$

Divide both sides by 4

$$\frac{6}{4} = \frac{4k}{4}$$

$$k = \frac{3}{2}$$

After finding the value of k , we then substitute it in the equation as shown below:

$$\text{The equation becomes: } y = \frac{x^2 k}{z} = \frac{3x^2}{2z}$$

To find the value of y , we substitute the values of x , z and k into the

$$\text{equation; } y = \frac{x^2 k}{z} = \frac{3x^2}{2z}$$

$$y = \frac{3 \times 3^2}{2 \times 2}$$

$$y = \frac{27}{4}$$

$$y = 6\frac{3}{4} \text{ or } y = 6.75.$$



This is the second activity. It is to help you work with different situations of joint variations.

Activity 2

Given that y varies as the square root of x and inversely as the square of z . When $y = 2$, $x = 9$ and $z = 3$, find the constant of variation. Find y when $x = 16$ and $z = 2$.



You have to compare your work with the work below in the feedback.

Feedback

When you checked the feedback to the first activity, it is hoped that you did check your weakness also. From these weaknesses, you have checked where to concentrate as you work through this activity. As in the other activity above, you have to translate the language into the mathematical one to make your work easy.

First translate into the language of variation

$$y \propto \sqrt{x} \text{ and } y \propto \frac{1}{z^2}$$

Then joining the two, we have the following statement: $y \propto \frac{\sqrt{x}}{z^2}$

$$\text{The equation is } y = \frac{k\sqrt{x}}{z^2}$$

To find the value of k by substituting y with 2, x with 9 and z with 3.

$$y = \frac{k\sqrt{x}}{z^2}$$

$$2 = \frac{k\sqrt{9}}{3^2}$$

Cross multiply: 2×9

$$2 = \frac{3k}{9}$$

$$18 = 3k$$

$$\frac{3k}{3} = \frac{18}{3}$$

Divide both sides by 3

$$k = 6$$

The equation becomes; $y = \frac{6\sqrt{x}}{z^2}$

To find the value of y , you substitute the values of x , z and k into the

equation: $y = \frac{k\sqrt{x}}{z^2}$

$$y = 6 \times \frac{\sqrt{16}}{2^2}$$

$$y = \frac{6 \times 4}{4}$$

$$y = \frac{24}{4}$$

$$y = 6$$

You have worked through the section on joint variation. You have learnt that in joint variation, three variables are connected together. One variable depends on two other variables. You should notice that you have not been introduced to the graphs in joint variation. The graphs here depend on three variables meaning that we should have a three axes coordinate plane as opposed to the two axes coordinate plane. It is beyond your syllabus for this type of graph. You will meet these graphs in mathematics in higher level colleges and universities.

The next part is about partial variation.

Partial Variation

We have so far discussed direct variation, inverse variation and joint variation. In joint variation, you learnt that a variable will depend on two or more other variables either both directly or one

directly and the other inversely. In partial variation, one variable will depend on just one other variable plus a constant. This type of dependence is partially. This will be explained further for clarity in the situations below.

In joint variation, there will be three variables being discussed while in partial variation, there will be only two variables being discussed. Let us consider the following situation to illustrate the above point.

A sales personnel at a computer store is paid a monthly salary of \$300 plus a 4% commission on monthly sales. The table below shows the income for monthly sales up to \$60 000.

Monthly Sales (S)	Monthly Income (I)
0	300
10 000	700
20 000	1 100
30 000	1 500
60 000	2 700

Table 6: Showing the monthly salary as compared to monthly sales

Table 6 does not represent a direct variation, though both sides show an increase. If the sales are doubled, the income is not doubled as shown in row 4 and row 5 of the table above. However, from the information in the table, we can express I (income) as a function of S (sales), as:

$I = 300 + 4\%$ of S, where the 4% is the commission (which is the changing variable depending on the sales) and the \$300 is the basic salary (which is the constant, not depending on any sales).

This means that income for the sales personnel is partially dependent on monthly sales. It implies that the officer will only get \$300 as income for that month if there are no sales. However, when there are sales, the income would change depending on the amount of sales for that particular month.

This situation is referred to as a **PARTIAL VARIATION**.

In general, we would say that in a partial variation situation, if y varies partially as x, then y can be expressed as a function of x. It is expressed in the form $y = mx + b$, where b is the constant.

We are going to have to work out two examples. Each of these examples will address a particular situation that may arise in partial variations. The first example will consider a one variable and constant situation while the second will consider the one variable situation.

The following example is designed to help you master the above principle and be able to practice the process of coming up with the equation and then solving it. Most Zambians know the dollar currency. Most of the people that access the internet also can access information on international currencies.

Example 3

A graduation dance costs \$5 000 plus \$20 for each person attending,

- Write an equation expressing the cost as a function of the number of people attending.
- Find the cost for 400 people.
- Draw the graph of the function.

Solution

We need to write down all the values given to help us form the equation.

- Let T be the total cost,

Let F be the fixed cost, \$5000

Let N be the number of people attending

The total cost of the dance is the sum of the fixed cost plus the cost of the number of people multiplied by the cost per person. This is expressed as:

$$T = F \text{ (fixed cost)} + 20 \times N \text{ (number of people)}$$

$$T = F + 20N$$

Therefore, the equation is: $T = 5\,000 + 20N$

- The cost for 400 people attending

$$T = 5\,000 + 20N$$

$$N = 400$$

$$T = 5\,000 + 20(400)$$

$$T = 5\,000 + 8\,000$$

$$T = 13\,000$$

Therefore, the total cost for 400 people is \$13 000.

- (c) The graph of $T = 5\,000 + 20N$ will be a straight line graph. We need to have a few points to help us draw the graph. The graduation cannot be attended by 1 person. We will take the minimum number of people attending to be 10. Here is the table of values;

N	10	20	30	40	50
T	5200	5400	5600	5800	6000

Table 7: Showing the values of N and T

We should remember that when $N = 0$, the cost of graduation is \$5000. This is the starting point.

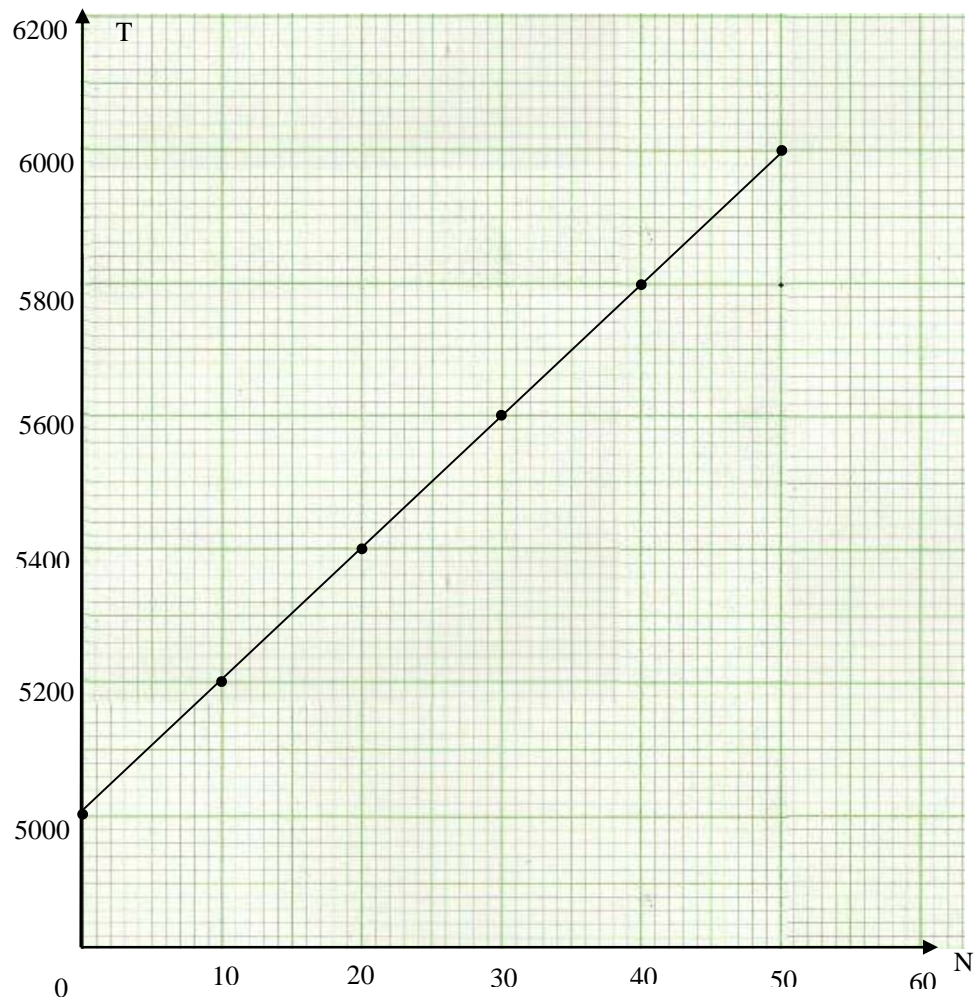


Figure 3: Showing the relationship between N and T

You might have noticed that the above example dealt with a variable depending upon one other variable and a constant. In this example, you learnt that the absence of people at the graduation would not make the cost of the graduation less expensive.

We will consider the other situation under partial variations.

The next example is to help you be able to find the constant of variation apart from just calculating the other variables

Example 4

The cost of the graduation ceremony is estimated at \$900 if 150 people attend and \$1020 if 250 people are to attend. The cost is made up of two items, one which is constant and the other directly proportional to the number of people attending. Find

- (a) The overhead.
- (b) The cost per person not counting the overhead.
- (c) The minimum charge per person at which graduation fee can be charged if there is a guaranteed attending of 300 people.

Solution

We need to write down all the values given to help us form the equation.

Let the total expense be T dollars

Let the fixed overhead be F

Let the number attending be N

Let k be the constant of variation in the part of the expenses which is variable.

We have to first find the value of k.

From the information given, then the formula is: $T = F + kN$.

Using the information above, we form two equations:

$$900 = F + 150k \text{ ----- (i)}$$

$$1020 = F + 250k \text{ ----- (ii)}$$

Using the same data, we solve simultaneously eq. (i) and (ii):

$$\begin{array}{r} 1020 = F + 250k \\ -(900 = F + 150k) \\ \hline 120 = 0 + 100k \\ 120 = 100k \end{array}$$

Divide both sides by 100

$$k = 1.2$$

Now having found the value of k, we have to find the following:

(a) Overhead or fixed expense

Substitute $k = 1.2$ into eq. (i) or (ii) above. We will use eq.(i):

$$900 = F + 150k$$

$$900 = F + 150(1.2)$$

$$900 = F + 180$$

$$F = 900 - 180$$

$$F = 720$$

Overhead or fixed expenses = \$720.

(b) You should note that the variable cost person is actually the constant of variation. We have already calculated above before attempting (a). Therefore, the variable cost per person is \$1.20.

(c) Minimum charge

$$\text{Formula: } T = F + 1.2 N$$

$$N = 300, F = 720$$

$$T = 720 + 1.2(300)$$

$$T = 720 + 360$$

$$T = 1080$$

Total expense for 300 people is \$1 080.

$$\text{Minimum charge} = \frac{T}{N}$$

$$\text{Minimum charge} = \frac{1080}{300}$$

$$\text{Minimum charge} = 3.6$$

Therefore, the minimum charge for the graduation fee if all 300 are to attend is \$3.60

Now, having done the examples, you should do the activity below to enable you understand the procedure of working out such problems.



It is similar to both examples above. It is, however, a different situation of partial variation, where one variable is partly dependent on one other variable. It is also to help you learn how to deal with such kinds of problems. Work out this on a separate sheet of paper. Then compare your answers with those provided in the feedback below.

Activity 3

The resistance to a car is partly proportional to its speed and partly proportional to the square of its speed. When the speed is 20km/h, the resistance is 60 N; when the speed is 30km/h, the resistance is 120N. Find the resistance when the speed is 40km/h.



You have to compare your work with the work below in the feedback.

Feedback

From the two examples above under partial variation, you might have noticed that there are only two variables for discussion. However there is also a constant which should never be left out unless otherwise. The constant is being considered as the fixed amount or quantity.

Now, in this activity, the constant is not explicit (clear). In other words, it is not shown as in the other activities and examples above. To form the equation in this situation, we now depend entirely on the statement given in the question. Notice that the statement states that resistance is partly on the speed and partly on the square of its speed. This means that resistance varies directly with both types of speeds.

We have to form equations first to help us work out this problem.

Let the resistance be R .

Let Speed be v km/h.

Here we add the two: that is, its normal speed and its square of speed to make the complete relationship of resistance and speed of the car.

Then the $R = av + bv^2$, where a and b are constants. The introduction of the constants (a and b) is to ensure that the statement is complete in mathematical terms. The coefficients are definitely present. They can be of value 1 or more but not zero. If they are zero, it means that the resistance will be zero. Hence it does not make sense to talk about resistance in this case. Again we cannot write the specific number value of a and b because we do not know their values at this moment. In the next steps, we will calculate for the values of a and b .

We then form equations using the information given. These equations are to be solved simultaneously.

When $v = 20$, $R = 60$, then $60 = 20a + 400b$(i)

When $v = 30$, $R = 120$, then $120 = 30a + 900b$(ii)

You should have solved the equations simultaneously;

Multiply (i) by 3 and (ii) by 2:

$180 = 60a + 1200b$(iii)

$240 = 60a + 1800b$(iv)

Subtract (iii) from (iv)

$60 = 600b$ (divide both sides by 600)

$$b = \frac{1}{10}$$

Substituting $b = \frac{1}{10}$ into (i) or (ii)

$60 = 20a + 400b$

$60 = 20a + 400\left(\frac{1}{10}\right)$

$60 = 20a + 40$

$60 - 40 = 20a$

$20 = 20a$

$1 = a$.

We can see that the coefficients a and b are of the values: $a = 1$ and $b = \frac{1}{10}$. When we substitute these values in the equation:

$R = av + bv^2$, we get the following complete equation.

Therefore, the equation becomes: $R = v + \frac{1}{10}v^2$

To find the resistance when the speed is 40 km/h, we substitute the value of the speed into the equation:

Speed (v) = 40 km/h,

$$R = v + \frac{1}{10}v^2$$

$$R = 40 + \frac{1}{10}(40^2)$$

$$R = 40 + \frac{1}{10}(1600)$$

$$R = 40 + 160$$

$$R = 200$$

Therefore, the resistance is 200N when the speed is 40km/h.

We have dealt with partial variation. We have seen that in partial variation, a variable will depend on one other variable plus a constant. In this case, the situation would be written as $y = ax + b$ where the variables are x and y . However, partial variation can also be one variable depending on one other variable. In this situation, partial variation can be written as: $y = ax + bx^n$, where n would be any integer value.

This concludes the topic. From the introduction, you might have noticed that this is the last of the two topics, hence the end of the unit. You can now read the topic summary.

Topic 2 Summary



As you have now come to the end of this topic, we can now review what you have learnt so far. The topic outcomes stated that you should be able to solve problems involving joint and partial variations. You should be able to draw the graphs of the Partial Variation.

In this topic, you were introduced to the joint and the partial variation. You have seen that under joint variation one variable is dependent on two or more other variables. You also learnt how to connect the variables in variation as one statement. You also learnt how to solve situations involving joint variations. You further learnt about partial variation and how the variables concerned in the situation are connected together in a statement. You learnt how to solve problems involving partial variation and how to illustrate the relationship in partial variation in graph form. In the above statements, we have dealt with our topic outcomes as reflected at the beginning of this topic.

To assess yourself on how much you have learnt under partial and joint variation, you should attempt the topic 2 exercise. Do the topic 2 exercise in the assignment section before proceeding to the tutor marked assignment. As stated earlier under topic 1 summary, you need to answer all the questions without checking with the answers provided at the end of this unit. After you have attempted the questions, then you may consult with the answers given.

Unit Summary



I hope that you were able to see or understand the relationship that you have by now seen the relationship of what you had learnt under ratios, proportion and rate in unit 5 to variation. You might have noted the usefulness of the concepts you learnt in unit 5 on ratio and proportions to this unit on variation. You did not do something completely different. It was the same but on a higher level. In this unit, we tried to answer the question, ‘what kind of relations exists between two variables?’ So your knowledge of ratios, proportion and rate played an important role.

In this unit, there were two topics. In topic 1, you learnt about direct and inverse variation. You learnt how to calculate the constant of variation and how to find the missing variable.

This topic addressed the first seven of the nine unit outcomes. The first was for you to learn how to express a given variation mathematically using symbols. The second was for you to learn how to change a given variation expressed mathematically into an equation and how to distinguish between inverse and direct variation. You went further to learn how to solve problems on direct and inverse variation and how to draw graphs of direct and inverse variation. Finally in the topic, you learn how to interpret graphs of direct and inverse variation.

In topic 2, you learnt about joint and partial variations. You learnt how to connect the two variations, that is; direct and inverse variation as joint variation. You also learnt how to use the partial variation to solve every day situational problems. This topic addressed the last two of the nine unit outcomes. The first outcome was that you should be able to solve problems based on joint variation. The second and last outcome was that you should be able to solve problems on partial variation.

By now you would have also completed the topic 2 exercise. This means that, besides the activities that you did within the topics, you have assessed your progress at the end of each of the two topics.

Congratulations on completing the sixth unit of the mathematics 11 course. There is a tutor marked assignment in this unit. We trust that the tutor marked assignment in addition to the activities and topic exercises will adequately help you to assess your own progress. You should complete and submit a tutor marked assignment now. This is a second TMA you are attempting. Remember that it is based on the knowledge from unit 4, 5 and 6.

The next unit, unit 7 is on travel graphs. You will learn how to draw and interpret distance-time graphs. You will also learn how to calculate the distance covered by an object.

References



1. Brendan Kelly and et al,(1990) *Mathematics II*, Addison-Sydney: Wesley Publishers Ltd,
2. Clarke H.L (1974), *Mathematics Four-Metric Edition*. London: Heinemann educational Books,
3. Dean J.E and Moore G.E (1964), *Essentials of Mathematics for Secondary Schools-BK 2*, Rinehart and Winston of Canada. Toronto.
4. Dudley B and Fyfield J A(1992), *The School Mathematics Dictionary*. London: Collins educational,
5. Rayner, D (1988), *General Mathematics: Revision and Practice*. Oxford: Oxford University Press,

Assignment



There are two exercises to be done in this unit. You have to do the assignment as instructed in the content section of this unit at the end of each topic. These are meant to help assess you on how much knowledge you have acquired in each part of this unit. It will not help you to skip these exercises. Attempt each one on a separate sheet of paper.

Afterwards, compare your answers with those provided at the end of the unit. Take care to work it out on your own. Remember that the work in exercise 1 depends on the knowledge of direct and inverse variation. Do not check the feedback before you work through the exercise. Some of the questions in this exercise are from Rayner D's Book entitled General mathematics-revision and Practice and Mukuyamba A and et al 's book entitled Mathematics Grade 11 Pupils book.

The work in exercise 2 depends on the knowledge of joint and partial variation. Do not check the feedback before you work through the exercise. Some of the questions in this exercise are from Mathematics Fourth-metric edition by Clarke H.L, Essentials of Mathematics for Secondary School-Book 2 by Dean J.E and Moore G. E and Mathematics 11 by Brendan Kelly and others.

Exercise 1

1. y varies directly as x and $y = 12$ when $x = 36$. Find the value of the constant of variation and find y when $x = 24$.
2. p is Proportional to q . when $p = 30$, $q = 6$. Find the value of the constant of variation and find p when $q = 12$.
3. y varies directly as the square of x and $y = 27$ when $x = 3$.
Find the value of y when $x = \frac{1}{3}$.
4. A varies as the square root of B and $A = 5$ when $b = 100$.
find
 - (i) A when $B = 49$
 - (ii) B when $A = 15$
5. Given that c varies directly as the square of x and $d = 36$ when $d = 4$ find:
 - (i) c when $d = 9$
 - (ii)** d when $c = 72$

6. y varies inversely as x and $y = 10$ when $x = 4$.
Find the value of the constant of variation and
find y when $x = 6$.
7. Given that p varies inversely as q . when $p = 6$, $q = 12$
Find the value of k
(a) Find p when $q = 24$
(b) Find q when $p = 8$
8. y varies inversely as the square of x , when $y = 4$, $x = 2$.
find the value of y when $x = 4$
9. Given that y varies inversely as \sqrt{x} when $y = 10$, $x = 9$
(i) Find the value of y when $x = 36$;
(ii) Find the value of x when $y = 30$
10. If a varies inversely as b^2 and $a = 24$
When $b = 2$ find
(a) a when $b = 2$
(b) b when $a = b$.
11. The resistance, R , of motion of a car is proportional to the
square of its speed, v , of the car. The resistance is 4000
newtons at a speed of 20 m/s.
(a) Find the resistance at a speed of 30 m/s.
(b) Find speed is the resistance 6250 newtons.
(c) Show this relationship by drawing the graph for v
values from 0 to 30 and R values from 0 to 9000. Take
the scale of 2 cm to 1000 units on the vertical axis and 2
cm to 5 units on the horizontal axis.
12. The number of hours N required to dig a certain hole is
inversely proportional to the number of men available x .
When 6 men are digging, the hole takes 4 hours.
(a) Find the time taken when 8 men are available.
(b) If it takes $\frac{1}{2}$ hour to dig the hole, how many men are
there?

- (c) Show this relationship by drawing the graph for x values from 0 to 12 and the N values from 0 to 12. Take the scale of 2 cm to 2 units on each axis.

Exercise 2

1. If a varies directly as b and c and $a = 6$ when $b = 4$ and $c = 9$, find the value of:
 - i. (i) a when $b = 3$ and $c = 10$.
 - ii. (ii) c when $a = 20$ and $b = 15$.
2. If p varies directly as q and inversely as v and $p = 35$ when $q = 7$ and $v = 6$, find the value of:
 - i. (i) p when $q = 3$ and $v = 16$
 - ii. (ii) v when $p = 15$ and $q = 4$.
3. If y varies as the square of x and inversely as z . when $y = 3$, $x = 2$ and $z = 1$. find y when $x = 4$ and $z = 2$.
4. If a varies directly as b^2 and inversely as the square root of c . when $a = 8$ when $b = 4$ and $c = 9$. Find the value of a when $b = 3$ and $c = 16$.
5. y varies directly as x and z . When $y = 4$, $x = 2$ and $z = 1$. Find y when $x = 8$ and $z = 4$.
6. An airplane at an altitude of 10 000 m begins to descend at 300 m/min.
 - i. Draw the graph showing the altitude of the plane as a function of time for the first 10 minutes of descend.
 - ii. Form an equation relating h metres to t minutes.

- iii. Using the equation, find the time taken in minutes of the plane at 4 000m.
7. The temperature of the earth's crust T degrees Celsius is a function of the depth d kilometres below the surface, where $T = 10d + 20$.
 - i. Graph the function for values of d up to 5Km
 - ii. The deepest mine is in South Africa, and reaches 3.8 km below the surface. Use the equation to find the approximate temperature at the bottom.
8. A motorist reckons that his annual expenditure is partly constant and partly varies as the number of kilometres travelled. When he travels 6000 km in a year the cost is £80; when he travels 7000 km in a year the cost is £90. Find the cost in a year when he travels 10 000km.
9. The cost of sending the football team to play an out-of-town game is partly constant and partly dependent upon the number of players who make the trip. If the cost of taking 18 players is \$855, and the cost for 26 players is \$ 1035, what should be the cost for 23 players?
10. The cost of printing the magazine is the sum of two quantities, one of which is fixed and the other one is directly proportional to the number of copies printed. The total cost for an issue of 200 copies is \$ 560. The cost for 500 copies is \$680. At what price can a copy be sold if there is a guaranteed sale of 700 copies?

Answers to Exercises in the Assignment



When you have done the work above, you should compare your answers with these below. The work may be very different from your work but the final answer is should be the same. This means that in mathematics, there are several ways in which a problem can be solved and not all can be written down. Feel free to use any other method you may have come across.

Exercise 1

1. $y \propto x$

$$y = kx$$

$$12 = 36k$$

$$\frac{1}{3} = k$$

∴ Value of the constant of variation is $\frac{1}{3}$

$$y = \frac{1}{3}k$$

$$y = \frac{1}{3} \times 24$$

$$y = 8.$$

2. $p \propto q$

$$p = kq$$

$$30 = 6k$$

$$5 = k$$

$$p = 5q$$

$$p = 5 \times 12$$

$$P = 60.$$

3. $y \propto x^2$

$$y = kx^2$$

$$27 = k3^2$$

$$27 = 9k$$

$$3 = k$$

$$y = 3x^2$$

$$y = 3 \times \left(\frac{1}{3}\right)^2$$

$$y = 3 \times \frac{1}{9}$$

$$y = \frac{1}{3}$$

4. $a \propto \sqrt{B}$

$$a = k\sqrt{B}$$

$$5 = k \times \sqrt{100}$$

$$5 = k \times 10$$

$$\frac{5}{10} = \frac{10k}{10}$$

$$\frac{1}{2} = k$$

(i) $a = \frac{1}{2} \times \sqrt{49}$

$$a = \frac{1}{2} \times 7$$

$$a = \frac{7}{2} \text{ or } a = 3\frac{1}{2}$$

(ii) $15 = \frac{1}{2} \times \sqrt{B}$

$$15 = \frac{\sqrt{B}}{2}$$

$$\sqrt{B} = 30$$

$$(\sqrt{B})^2 = 30^2$$

$$B = 900$$

5. $c \propto \sqrt{d}$

$$c = k\sqrt{d}$$

$$36 = k \times \sqrt{4}$$

$$36 = 2k$$

$$18 = k$$

$$(i) \quad c = 18 \times \sqrt{9}$$

$$c = 54$$

$$(ii) \quad \frac{72}{18} = \frac{18\sqrt{d}}{18}$$

$$4 = \sqrt{d}$$

$$16 = d$$

$$6. \quad y \propto \frac{1}{x}$$

$$y = \frac{k}{x}$$

$$10 = \frac{k}{4}$$

$$k = 40$$

$$y = \frac{40}{x}$$

$$y = \frac{40}{6}$$

$$y = \frac{20}{3} \text{ or } y = 6\frac{2}{3}$$

$$7. \quad p \propto \frac{1}{q}$$

$$p = \frac{k}{q}$$

$$6 = \frac{k}{12}$$

$$k = 72$$

$$p = \frac{72}{q}$$

$$(i) \quad p = \frac{72}{24}$$

$$p = 3$$

$$(iii) \quad 8 = \frac{72}{q}$$

$$8q = 72$$

$$q = 9.$$

$$8. \quad y \propto \frac{1}{x^2}$$

$$y = \frac{k}{x^2}$$

$$4 = \frac{k}{2^2}$$

$$4 \times 4 = k$$

$$k = 16$$

$$y = \frac{16}{x^2}$$

$$y = \frac{16}{4^2}$$

$$y = \frac{16}{16}$$

$$y = 1.$$

$$9. \quad y \propto \frac{1}{\sqrt{x}}$$

$$y = \frac{k}{\sqrt{x}}$$

$$10 = \frac{k}{\sqrt{9}}$$

$$10 = \frac{k}{3}$$

$$k = 30$$

$$y = \frac{30}{\sqrt{x}}$$

$$(i) \quad y = \frac{30}{\sqrt{36}}$$

$$y = \frac{30}{6}$$

$$y = 5.$$

$$(ii) \quad y = \frac{30}{\sqrt{x}}$$

$$30 = \frac{30}{\sqrt{x}}$$

$$\frac{30\sqrt{x}}{30} = \frac{30}{30}$$

$$\sqrt{x} = 1$$

$$(\sqrt{x})^2 = 1^2$$

$$\therefore \mathbf{x = 1}$$

$$10. \quad a \propto \frac{1}{b^2}$$

$$a = \frac{k}{b^2}$$

$$24 = \frac{k}{2^2}$$

$$24 = \frac{k}{4}$$

$$K = 96$$

$$(i) \quad a = \frac{96}{2^2}$$

$$a = \frac{96}{6}$$

$$\mathbf{a = 24}$$

$$(ii) \quad 6 = \frac{96}{b^2}$$

$$\frac{6b^2}{6} = \frac{96}{6}$$

$$b^2 = 16$$

$$\sqrt{b^2} = \sqrt{16}$$

$$b = 4.$$

You find the square root of both sides

11. The relationship is $R \propto V^2$

(a) $R = kV^2$

When $R = 4000$, $V = 20$

We use these values to find the value of k

$$4000 = k \times 20^2$$

$$4000 = 400k$$

$$10 = k$$

The equation becomes: $R = 10V^2$

When $v = 30$, $R = 10 \times 30^2 = 10 \times 900 = 9000$

The resistance is 9 000 newtons.

(b) To find for speed when $R = 6250$,

$$R = 10V^2$$

$$6250 = 10V^2$$

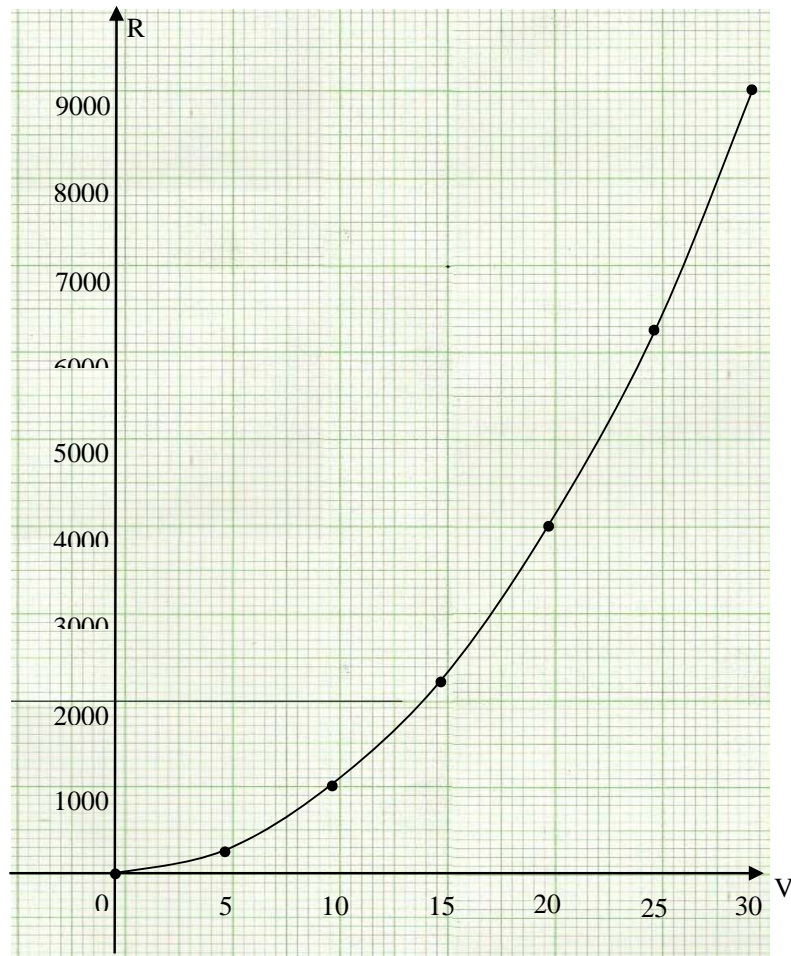
$$625 = V^2$$

$$\sqrt{625} = \sqrt{V^2}$$

$$25 = V$$

Therefore, the speed is 25 m/s.

(c) The graph after all the work is done will look like this:



12 . The relationship is $N \propto \frac{1}{x}$ and the equation is: $N = \frac{k}{x}$

(a) $N = \frac{k}{x}$

When $N = 4$, $x = 6$

We use these values to find the value of k

$$4 = \frac{k}{6}$$

$$4 \times 6 = k$$

$$24 = k$$

The equation becomes: $N = \frac{24}{x}$

$$\text{When } x = 8, N = \frac{24}{8} = 3$$

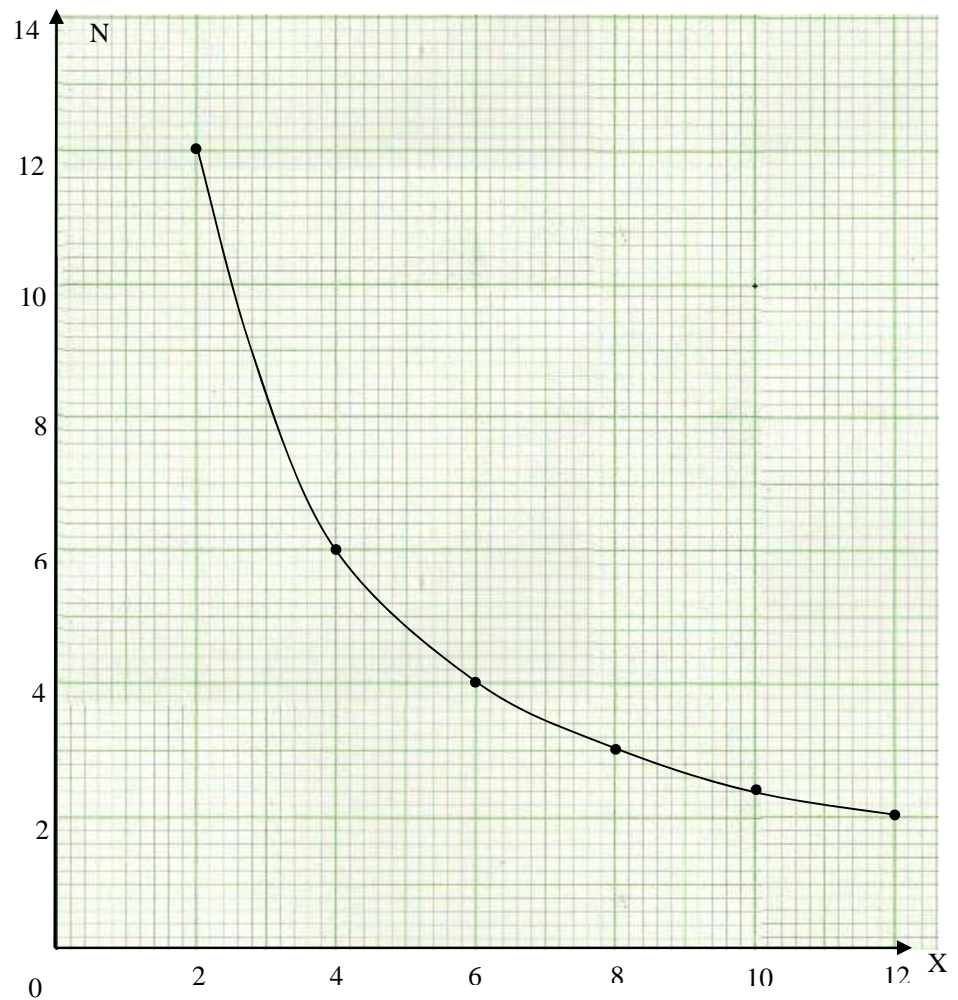
The number of hours required is 3 hours

(b) To find the number of men when $N = \frac{1}{2}$,

$$\begin{aligned}N &= \frac{24}{x} \\ \frac{1}{2} &= \frac{24}{x} \\ X &= 2 \times 24 \\ X &= 48\end{aligned}$$

Therefore, the number of men required is 48

(c) The graph after all the work is done will look like this:



Exercise 2

1. $a \propto b$ and $a \propto c$

$$a = kbc$$

$$6 = k \times 4 \times 9$$

$$\frac{6}{36} = \frac{\cancel{36}k}{\cancel{36}}$$

$$\frac{1}{6} = k$$

$$a = \frac{1}{6}bc$$

(i) $a = \frac{1}{6} \times 3 \times 10$

$$a = 5$$

(ii) $20 = \frac{1}{6} \times 15 \times c$

$$\frac{20}{1} = \frac{5c}{2}$$

$$5c = 40$$

$$\frac{5c}{5} = \frac{40}{5}$$

$$c = 8.$$

2. $p \propto q$ and $p \propto \frac{1}{r}$

$$p \propto \frac{q}{r}$$

$$p = \frac{kq}{r}$$

$$35 = \frac{7k}{6}$$

$$35 \times 6 = 7k$$

$$\frac{\cancel{7}k}{\cancel{7}} = \frac{6 \times 35}{7}$$

$$k = 30.$$

$$p = \frac{30q}{r}$$

$$(i) \quad p = \frac{30 \times 3}{16}$$

$$p = \frac{45}{8}$$

$$(ii) \quad 15 = \frac{30 \times 4}{r}$$

$$15r = 30 \times 4$$

$$\frac{\cancel{15}r}{\cancel{15}} = \frac{30 \times 4}{15}$$

$$r = 2 \times 4$$

$$\mathbf{r = 8.}$$

$$3. \quad y \propto x^2 \quad \text{and} \quad y \propto \frac{1}{z}$$

$$y \propto \frac{x^2}{z}$$

$$y = \frac{kx^2}{z}$$

$$3 = \frac{k \times 2^2}{1}$$

$$3 = \frac{4k}{1}$$

$$4k = 3$$

$$k = \frac{3}{4}$$

$$y = \frac{3x^2}{4z}$$

$$y = \frac{3 \times 4^2}{4 \times 2}$$

$$y = \frac{3 \times 16}{8}$$

$$y = 3 \times 2$$

$$\mathbf{y = 6}$$

$$4. \quad a \propto b^2 \text{ and } a \propto \frac{1}{\sqrt{c}}$$

$$a \propto \frac{b^2}{\sqrt{c}}$$

$$a = \frac{kb^2}{\sqrt{c}}$$

$$8 = \frac{k \times 4^2}{\sqrt{9}}$$

$$8 = \frac{16k}{3}$$

$$8 \times 3 = 16k$$

$$\frac{16k}{16} = \frac{24}{16}$$

$$k = \frac{3}{2}$$

$$a = \frac{3b^2}{2\sqrt{c}}$$

$$a = \frac{3 \times 3^2}{2\sqrt{16}}$$

$$a = \frac{3 \times 9}{2 \times 4}$$

$$a = \frac{27}{8} \text{ or } a = 3\frac{3}{8}$$

$$5. \quad y \propto x \text{ and } y \propto z$$

$$y \propto xz$$

$$y = kxz$$

$$4 = k \times 2 \times 1$$

$$\frac{4}{2} = \frac{2k}{2}$$

$$\mathbf{2 = k}$$

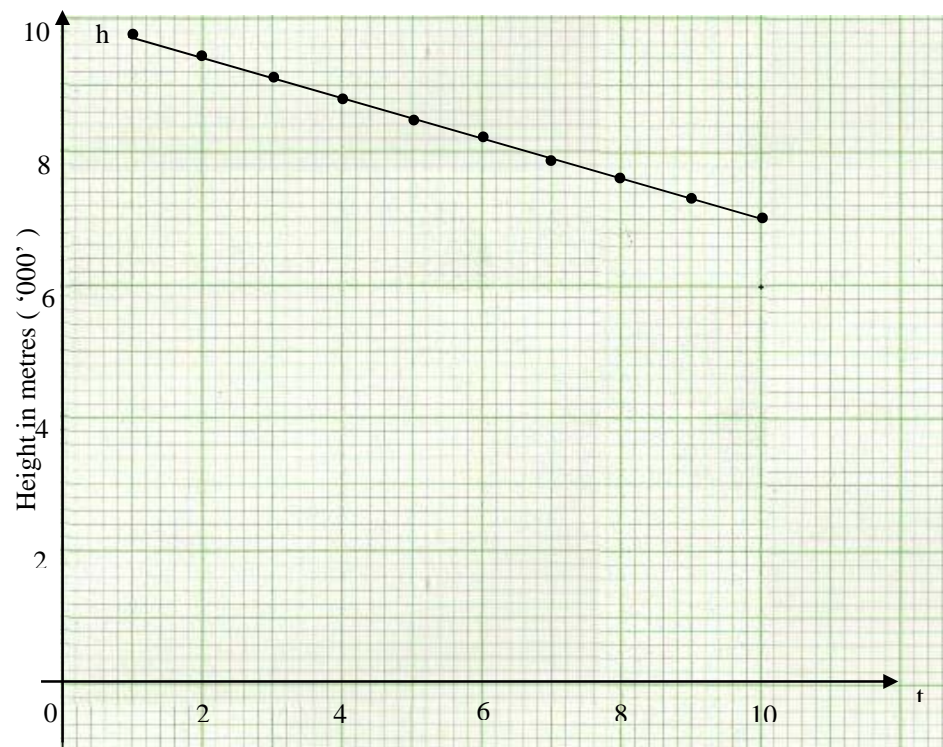
$$y = 2xz$$

$$y = 2 \times 8 \times 4$$

$$\mathbf{y = 64.}$$

6. Using the rate of 300 m per minutes, we make a table of values as shown below;

t	1	2	3	4	5	6	7	8	9	10
h '000'	9.7	9.4	9.1	8.8	8.5	8.2	7.9	7.6	7.3	7.0



- (b) since the airplane is descending, then the formula is:

$$h = 10\,000 - 300t.$$

- (c) since $h = 4000$ m.

$$h = 10\,000 - 300t.$$

$$4\,000 = 10\,000 - 300t$$

$$4\,000 - 10\,000 = -300t$$

$$-6\,000 = -300t$$

$$T = 20$$

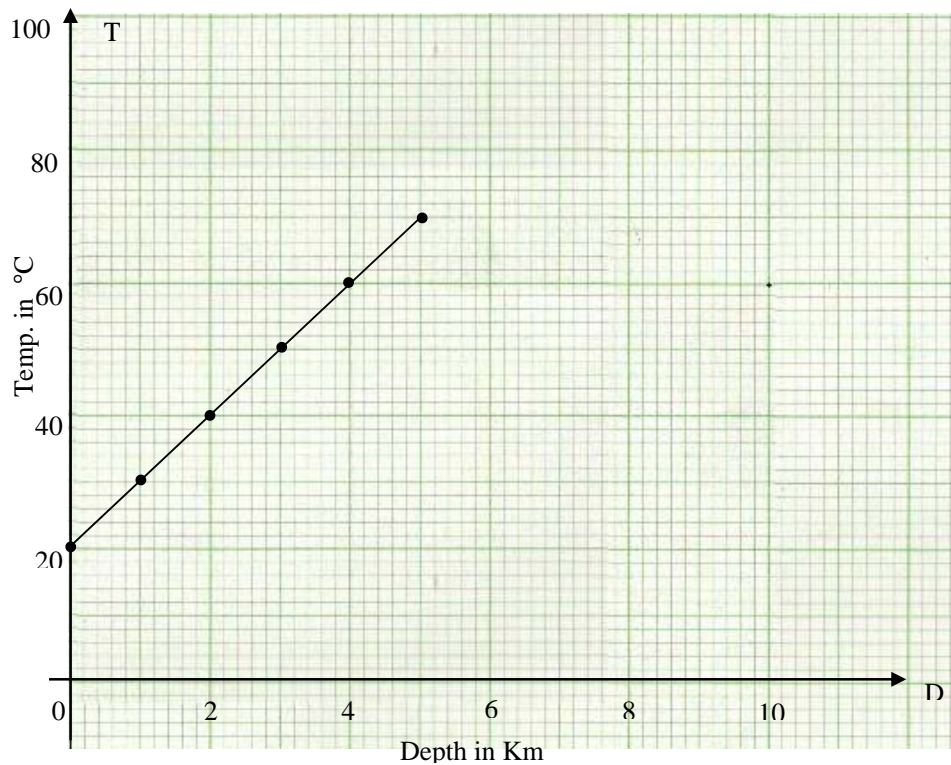
Therefore, the time taken is 20 minutes.

7. Using the distance of 1 to 5 km, we make a table as shown below;

(a) $T = 10d + 20$

Using the equation and the distance values given, the table of values is as follows:

D	1	2	3	4	5
T	30	40	50	60	70



When the Depth is zero ($D = 0$), the Temperature is at 20°C .

$$(b) T = 10d + 20$$

$$D = 3.8 \text{ km}$$

$$T = 10(3.8) + 20$$

$$T = 38 + 20$$

$$T = 58$$

The temperature is 58°C .

8. Let the number of kilometres be d
 Let F be the fixed expense
 Let E be the annual expenditure
 Let the constant of variation be k

The equation is $E = F + kd$

When $d = 6000$, $E = 80$ and when $d = 7000$, $E = 90$

$$F + 6000k = 80 \dots \dots \dots \text{equ. (i)}$$

$$F + 7000k = 90 \dots \text{equ. (ii)}$$

Subtract equ.(i) from equ.(ii)

$$1000k = 10$$

$$K = \frac{10}{1000}$$

$$K = \frac{1}{100}$$

Substituting $k = \frac{1}{100}$ into (i) or (ii)

$$F + 7000 \times \frac{1}{100} = 90$$

$$F + 70 = 90$$

$$F = 90 - 70$$

$$F = 20$$

The fixed expense is £ 20.

The equation becomes: $E = 20 + \frac{1}{100}d$

When $d = 10\,000$

$$E = 20 + \frac{1}{100}d$$

$$E = 20 + \frac{1}{100} \times 10\,000$$

$$E = 20 + 100$$

$$E = 120$$

Therefore, annual expense is £120

9. Let the cost be C
 Let F be the fixed expense
 Let N be the annual expenditure
 Let the constant of variation be k

The equation is $C = F + kN$

When $N = 200$, $C = 560$ and when $N = 500$, $C = 680$

$$F + 200k = 560 \dots \text{equ. (i)}$$

$$F + 500k = 680 \dots \text{equ. (ii)}$$

Subtract equ.(i) from equ.(ii)

$$300k = 120$$

$$K = \frac{120}{300}$$

$$K = \frac{2}{5}$$

Substituting $k = \frac{2}{5}$ into (i) or (ii)

$$F + 500 \times \frac{2}{5} = 680$$

$$F + 200 = 680$$

$$F = 680 - 200$$

$$F = 480$$

The fixed expense is \$480.

The equation becomes: $C = 480 + \frac{2}{5}N$

When $N = 700$

$$C = 480 + \frac{2}{5}N$$

$$C = 480 + \frac{2}{5} \times 700$$

$$C = 480 + 280$$

$$C = 760$$

Therefore, the Cost is \$760

10. Let the cost be C

Let F be the fixed expense

Let N be the annual expenditure

Let the constant of variation be k

The equation is $C = F + kN$

When $C = 855$, $N = 18$ and when $C = 1035$, $N = 26$

$$F + 18k = 855 \dots \dots \dots \text{equ.(i)}$$

$$F + 26k = 1035 \dots \dots \dots \text{equ.(ii)}$$

Subtract equ.(i) from equ.(ii)

$$8k = 180$$

$$K = \frac{180}{8}$$

$$K = 22.5$$

Substituting $k = 22.5$ into (i) or (ii)

$$F + 18 \times 22.5 = 855$$

$$F + 405 = 855$$

$$F = 855 - 405$$

$$F = 450$$

The fixed expense is \$ 450.

The equation becomes: $C = 450 + 22.5N$

When $N = 23$

$$C = 450 + 22.5d$$

$$C = 450 + 22.5 \times 23$$

$$C = 450 + 517.5$$

$$C = 967.5$$

Therefore, cost is \$967.5

We hope that after comparing your answers with the model answers provided in the above feedback you might have got all the answers correct. Congratulations! If not then we suggest you try the questions you got wrong again. You may be required to read through the section again in order to gain more understanding.

Assessment



This assessment is based on the 3 units. It is based on unit 4 on solving quadratic equations, unit 5 on ratios, proportions and rate and finally on unit 6 on variation. You need the knowledge from all these units to attempt these questions. There are only 6 questions in this assessment. Attempt all and show your working on the separate answer sheet you are to use.

Questions in this assessment are taken from the following books. Questions 1 and 2 are from Mathematics Grade 11 Pupils' book by Mukuyamba A and others, and questions 3, 4 and 5 from New General Mathematics book 3 by Channon J B and others.

1. The pressure, P , on a disc immersed in a liquid varies as the depth, d , and as the square of the radius, r , of the disc. The pressure is 2 500 Pa [Newton's per square metre] when the depth is 4m and the radius is 3m.
 - (a) Find a formula for P
 - (b) Calculate
 - (i) P when the depth is 3m and the radius is 2.8m
 - (ii) D , when pressure is 5 000 Pa and radius 4m.
2. If y varies directly as x and z and $x = 6$ when $y = 4$ and $z = 9$, find the value of
 - (a) x when $y = 3$ and $z = 10$
 - (b) z when $x = 20$ and $y = 15$
3. A rectangular plot measures 12m by 5m. A path of constant width runs along one side and one end. If the total area of the plot and the path is 120 m^2 , find the width of the path.
4. A metal is composed of copper and zinc in the ratio 3: 2 by volume. Find the volume of a piece of the metal which contains 42 cm^3 of copper.
5. In a town with a population of 53 280, there were 562 deaths in one year. Find the death rate per 1000 persons correct to 3 s.f..
6. The Ministry of Health held a training workshop for nurses. Each nurse was expected to pay an amount of K 250 000 towards the training. The total cost of the workshop was estimated at K 14 500 000 plus K 250 000 for each person attending.
 - (a) Write an equation expressing the cost as a function of the number of people attending
 - (b) Find the cost for 56 nurses
 - (c) Draw the graph of the function. Take the scale of 2cm to 5 000 000 on the cost axis for values from 5 000 000 to 30 000

000 and 2 cm to 10 persons on the number axis for values from 10 to 60.

Answers to the Assessment

$$1. \quad p \propto d \quad \text{and} \quad p \propto r^2$$

$$p \propto dr^2$$

$$p = kdr^2$$

$$2500 = 4 \times 3^2 \times k$$

$$2500 = 36k$$

$$\frac{2500}{36} = \frac{36k}{36}$$

$$k = \frac{2500}{36}$$

$$k = \frac{625}{9}$$

$$p = \frac{625}{9} dr^2$$

$$(i) \quad d = 3 \text{ m} \quad r = 2.8 \text{ m}$$

$$p = \frac{625}{9} dr^2$$

$$p = \frac{625}{9} \times 3 \times 2.8^2$$

$$p = \frac{625}{3} \times 7.84$$

$$p = \frac{4900}{3}$$

$$p = \mathbf{1633}$$

$$(ii) \quad p = \frac{625}{9} dr^2$$

$$5000 = \frac{625}{9} \times d \times 4^2$$

$$5000 = \frac{625}{9} \times 16d$$

$$d = \frac{45000}{16 \times 625}$$

$$d = \frac{45000}{10000}$$

$$d = \mathbf{4.5}$$

$$\begin{aligned}2. \quad & y \propto x \text{ and } y \propto z \\ & y \propto xz \\ & y = kxz \\ & y = 4, x = 6 \text{ and } z = 9 \\ & 4 = k \times 6 \times 9 \\ & 4 = k \times 54 \\ & k = \frac{4}{54}\end{aligned}$$

$$k = \frac{2}{27}$$

$$y = \frac{2}{27}xz$$

$$(a) \quad y = 3, z = 10$$

$$y = \frac{2}{27}xz$$

$$3 = \frac{2}{27} \times x \times 10$$

$$81 = 20x$$

$$x = \frac{81}{20}$$

$$x = 4.1$$

$$(b) \quad y = 15, x = 20$$

$$y = \frac{2}{27}xz$$

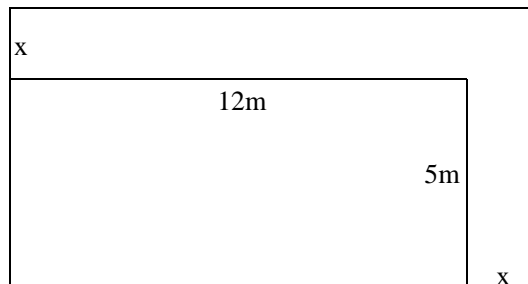
$$15 = \frac{2}{27} \times 20z$$

$$405 = 40z$$

$$z = \frac{405}{40}$$

$$z = 10.1$$

3. Sketch of the diagram



$$\text{Length} = (12 + x) \text{ m}$$

$$\text{Bread} = (5 + x) \text{ m}$$

$$\text{Area} = lb$$

$$\text{Area} = (12 + x)(5 + x)$$

$$\text{Area} = 60 + 12x + 5x + x^2$$

$$120 = x^2 + 17x + 60$$

$$x^2 + 17x + 60 - 120 = 0$$

$$x^2 + 17x - 60 = 0$$

$$x^2 - 3x + 20x - 60 = 0$$

$$x(x - 3) + 20(x - 3) = 0$$

$$(x - 3)(x + 20) = 0$$

$$\text{Either, } x - 3 = 0 \text{ or } x + 20 = 0$$

$$x - 3 = 0 \text{ or } x + 20 = 0$$

$$x = 3 \text{ or } x = -20$$

The negative value is rejected.

Therefore, $x = 3$ is taken.

The path is 3m width.

4. Ratio: 3:2

$$\text{Total parts: } 3 + 2 = 5$$

$$\text{Volume of copper in the metal} = 42 \text{ cm}^2$$

$$1 \text{ part of metal} = \frac{42}{3} = 14 \text{ cm}^2$$

The volume of metal = volume of 1 part \times number of parts

$$\text{The volume of metal} = 14 \text{ cm}^2 \times 5$$

$$\text{The volume of metal} = \mathbf{70 \text{ cm}^2}$$

$$5. \text{ Percentage of death} = \frac{\text{Number of death}}{\text{Total Population}} \times 100\%$$

$$\text{Percentage of death} = \frac{562}{53280} \times 100\%$$

$$\text{Percentage of death} = 0.01055 \times 100\%$$

Percentage of death = 1.055%

Death rate = percentage of death \times 1000

Death rate = 1.055% \times 1000

$$\text{Death rate} = \frac{1.055}{100} \times 1000$$

$$\text{Death rate} = \frac{1055}{100}$$

Death rate = **10.5**

6. Let the cost be C
Let F be the fixed expense
Let N be the number of participants

(a) The equation becomes: $C = 14\,500\,000 + 250\,000N$

(b) When $N = 56$

$$C = 14\,500\,000 + 250\,000N$$

$$C = 14\,500\,000 + 250\,000 \times 56$$

$$C = 14\,500\,000 + 14\,000\,000$$

$$C = 28\,500\,000$$

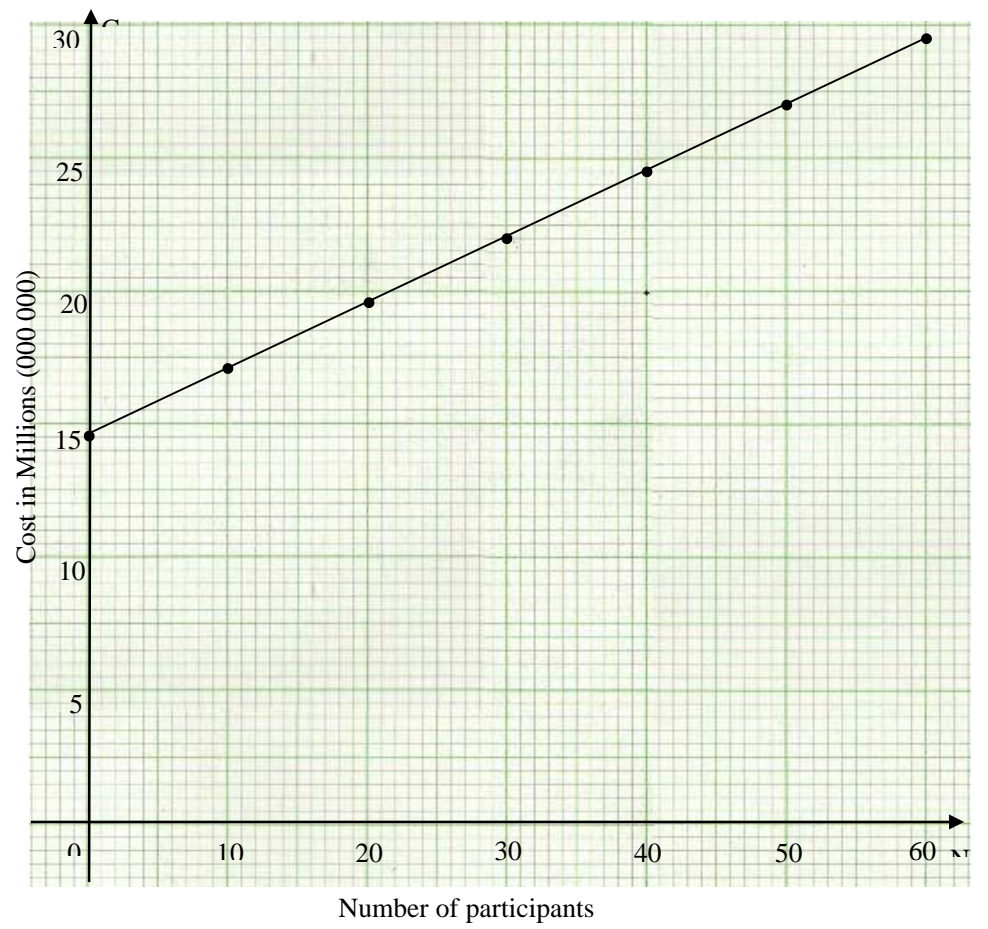
Therefore, cost is K 28 500 000

(c) The table of values is:

N	10	20	30	40	50	60
C	17	19.5	22	24.5	27	29.50

Note: values in C are in Millions

When the graph is drawn, it will look like this.



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Unit 7

Travel Graphs

Introduction

Welcome to unit 7 in your mathematics 11 course. In unit 6 you learned how to solve problems on direct variation, inverse variation and partial variation. You also learned how to write equations connecting direct variation, inverse variation and joint variation.

In this unit you will learn how to draw and interpret the distance-time graphs and the speed-time graphs. You will learn how to calculate the distance, speed, time and acceleration from the given graphs.

The travel graphs are similar to the graphs of polynomials you learned earlier in this course (in unit 3). What is similar in both units is the procedure of plotting the graphs, finding the gradient of the straight line and finding the area under the curve. In this unit, the gradient of the straight line we will calculate in the distance-time graphs represents the speed while in speed-time graphs the gradient represents the acceleration and retardation. The area under the curve represents the distance covered in a given time.

This unit has two topics namely distance-time graphs and speed-time graphs. In each topic, you will be required to do some activities and topic exercises. As in all previous units, you are encouraged to answer all the questions in the topic exercises before checking the answers provided in the feedback.

Upon completion of this unit you will be able to:

- *Draw and interpret* the distance-time graph and speed-time graph
- *Calculate* the distance covered in a specified time from a speed –time graph.



Outcomes

We hope you have carefully studied the unit outcomes? The outcomes give you an indication of the basic competencies you need to gain by working through this unit.

Timeframe



How long?

We estimate that to complete this unit you will need between 10 and 12 hours. This time includes the time you will spend in doing the activities and checking them against the feedback. If you do not finish studying the unit within this estimated time do not worry since we do not all learn at the same pace.

You are encouraged to spend about 2 hours answering each topic exercise in this unit. There are two exercises in this unit and you will spend about 4 hours on these exercises.

The total hours for completing the unit will thus be between 14 and 16 hours.

Learning resources



Resources

In order to study this unit with minimal difficulties you will need the following materials:

- A ruler
- A pencil or a pen
- A calculator
- Graph papers

Teaching and Learning Approaches



Tip

In this unit we have used three teaching and learning methods in presenting the content. These methods are:

- **Conceptual:** You will be introduced to concepts and be helped to understand the meaning of facts, rules, formulas, and procedures;
- **Problem-solving:** You will use the knowledge gained in conceptual methods to solve mathematical problems that relate to real life situations. You are encouraged to discuss mathematical problems, answers and strategies with friends and your tutor/s.

- **Skills:** You are encouraged to practice using the facts, rules, formulas and procedures by doing activities throughout the unit and topic exercises at the end of each topic.

As you read through the unit, do the activities and/or exercises and discuss your ideas with other learners and your tutor; you will be putting into practice these three teaching and learning methods. When you consistently apply these methods, we trust that you will achieve greater understanding of the unit and be able to relate the knowledge to your real life situation.

Below is a list of key concepts that will be covered in this unit content.



Terminology

Acceleration (or retardation):	The rate at which the velocity of a moving object changes with respect to time.
Cartesian:	Derived from Descartes, French mathematician. Cartesian coordinates form a reference system in which any point in a plane is located by its displacements from two fixed lines called axes.
Gradient:	The slope or steepness of a line or curve.
Ordered pair:	A pair of numbers for which their order is important.
Speed:	The rate at which a moving object changes position with respect to time.
Unit:	A defined amount of a quantity, serving as a basis for measuring other amounts of the same quantity, e.g. the metre is defined as a unit of length.
Velocity:	The rate of change of distance
Trapezium:	A quadrilateral with two parallel sides

(Source: John A. Fyfield and Dudley Blane (2002). *The School Mathematics Dictionary*; and Encarta Dictionaries)

Now work through the following topics which will help you to explain the above concepts and enable you to achieve the basic competencies given in the outcomes above.

Topic 1: Distance-Time Graphs

In this first topic of the unit travel graphs, you will learn how to calculate the speed of the moving object from the given distance-time graphs. You will also learn how to draw the distance-time graphs from the given information on the graph paper.

A distance-time graph is another interesting topic in your mathematics course. The knowledge on distance-time graph is important in our daily lives as we use it in various ways. For instance we use distance-time graphs to show the relationship between the distance covered by a moving object and time taken. We can as well use the distance –time graphs to compare the speed of more than one object.

As in other topics, we have included a number of activities for you to do in order to help you understand the topic better and apply the skills that have been taught. At the end of the topic there is topic exercise 1 for you to do. You are encouraged to answer all questions in the topic exercise and mark your own work by comparing your answers against those provided in the feedback that follows. You are encouraged not to go through the feedback before doing the topic exercises so that you can assess your understanding of each topic correctly.

In this topic, we will address the first of the two unit outcomes, that is:

- *Draw and interpret* the distance-time graph and speed-time graph

This outcome has been divided into objectives as listed below.

Upon completion of this topic you will be able to:



Objectives

- *Plot* coordinates of points on the graph paper.
- *Draw* the distance time graph on the graph paper.
- *Calculate* the speed from the distance-time graph.
- *Solve* problem on distance-time graph.

The objectives you have just gone through are met to help you gain an insight of what you ought to achieve at end the topic. The only way we could help you gain knowledge and skills based on the objectives we have just outlined is by taking you through a number of the activities. You will not have much difficulty understanding these activities since we will take you through them step by step. Let us begin our study on how to draw the distance-time graph.

Drawing the Distance-time Graphs

The relationship between distance travelled by a moving body or object and the time taken can be presented graphically.

On the graph, normally we represent the **distance travelled vertically** as y-axis and the **time taken horizontally** as x – axis as shown in the diagram included in the example below.

The following example will help you understand better how to draw the distance-time graph.

Example 1

A boy starts cycling his bicycle from school at 10:00 hours to his village. The distance between the school and his village is 20 km. He takes 2 hours to reach the village and stays there for 30 minutes. He then goes back to school and arrives there at 15:30 hrs.

Draw the distance-time graph to represent his journey.

As you learned in unit 3, in this example we will also use graph paper to draw the graph to represent this information.

Solution

To draw the graph, we have to follow the following steps:

Step 1: We have to come up with the appropriate scale. In this instance our scale will be 1cm or 1 square box to represent 5 km on the vertical axis and 1 cm or 1 square box to represent 30 minutes on horizontal axis. You should note that the first number for horizontal axis is the time the boy left his school.

Our two axes on the graph will be as shown below.

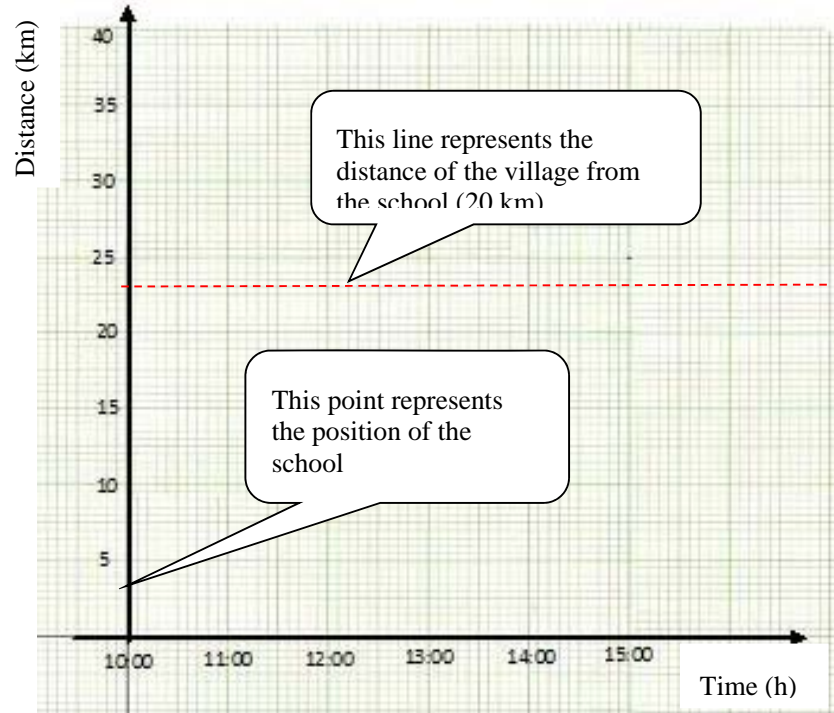


Figure 1: Diagram showing distance-time axes on Cartesian graph

Let us move to step two.

Step 2: We have to think of the time taken to cover a certain distance as an **ordered pair**. The term ordered pair is not new to you as you learned about it earlier in this course in unit 3. If you cannot remember its meaning clearly, then you can check its meaning in the terminology above.



Note it!

At a constant speed, for the first 2 hours the boy covered 20 km. We should as well note that 2 hours after 10:00 hours is 12:00 hours. Thus we plot 12:00 hours for vertical axis and 20 km for horizontal axis. Therefore, the point we will plot on the Cartesian graph is (12:00 hours, 20 km), as shown in the Figure 2.

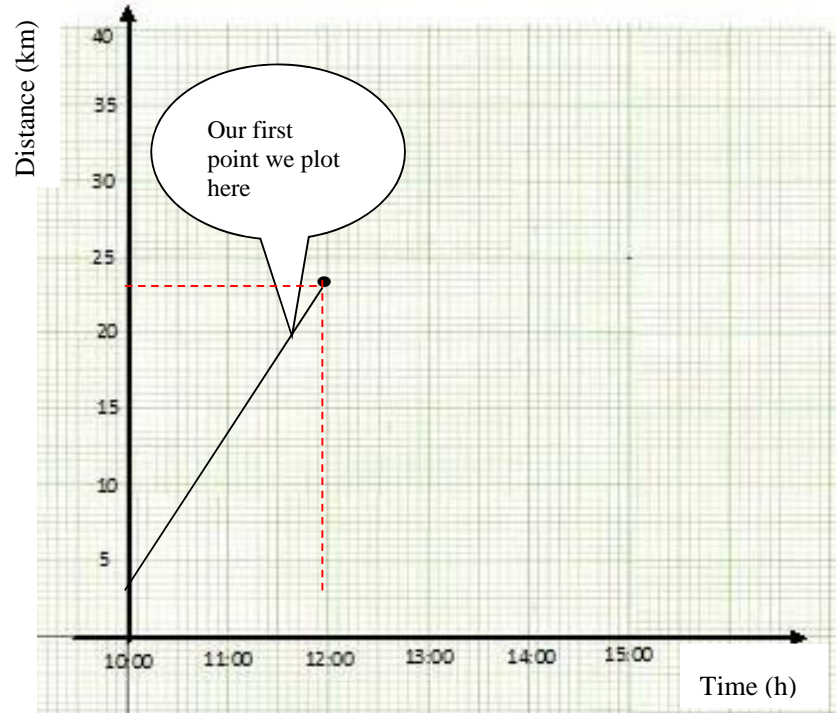


Figure 2: Distance-time graph describing the journey of a boy

Let us move to third step together.

Step 3: At this stage we will add to the graph the time the boy stayed at the village. Since he arrived at 12:00 hours and stayed 30 minutes then he returns to school at 12:30 hours.



Note it!

Since there is no change in the distance when the boy was at home, the line we will add to the previous one is horizontal.

Check on the graph now and take note of the horizontal line we are talking about.

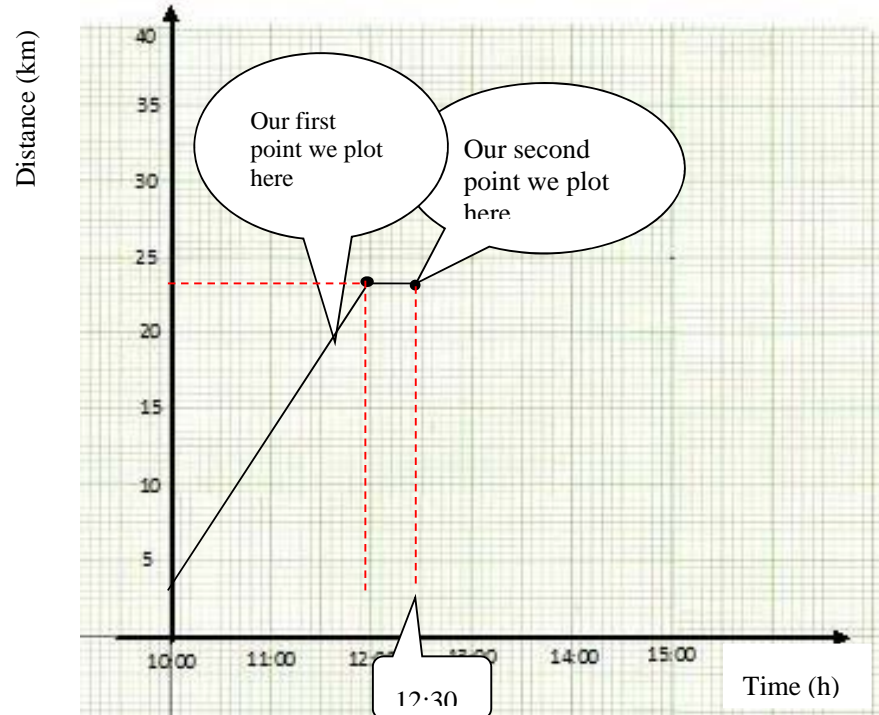


Figure 3: Distance-time graph describing the journey of a boy

Let us move to fourth step together.

Step 4: This is the last step. We will draw the last part of the graph. We will extend the line from the last point; (12:30 h, 20 km) to (15:30hours, 0 km).



Note it!

You should note that the 0 km represents the place where the boy started from. The graph below will be the culmination of the progressive development suggested above.

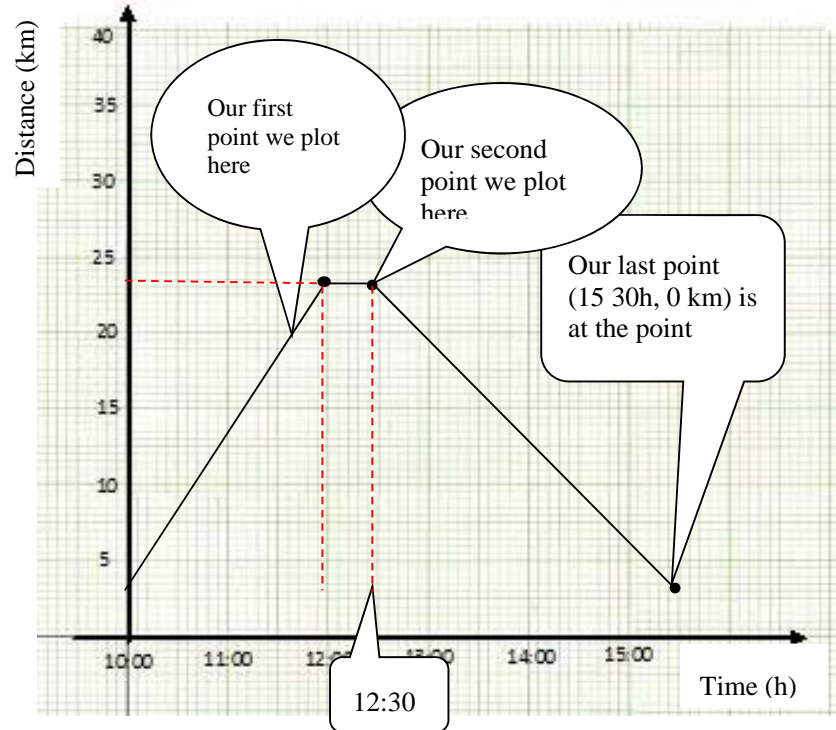


Figure 4: Distance-time graph describing the journey of a boy

Look at the graph above again and do the following activity.



Activity 1

Find the gradient of the lines that represent journeys the boy travelled from school to the village and back to school?

Write your answer in the space below.



Feedback

Feedback

You should note that the line representing the journey from school to the village slopes upwards and the line representing the returning journey slopes downwards.

*In unit 3 you learned that the steepness of the line is called the **gradient**.*

*The gradient of the line distance-time graph represents the **speed**.*

The formula for finding the gradient is given as

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}}$$

where the vertical change is y-axis and horizontal change is x-axis.

$$\text{Therefore, gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Or gradient} = \frac{y}{x}$$

Calculating the Speed of the Boy from School to the Village

You should note that in the distance-time graph above the vertical increase is 20 km and the horizontal increase for the same distance is 2 hours.

Therefore, the speed for the boy in his journey from school to his village is

$$\text{Gradient} = \frac{\text{distance travelled}}{\text{time taken}} \text{ or}$$

$$\text{Speed} = \frac{\text{distance travelled}}{\text{time taken}} ; \text{ where distance} = 20 \text{ km, time} = 2 \text{ hours}$$

$$\begin{aligned} \text{Speed} &= \frac{20\text{Km}}{2\text{hrs}} \\ &= 10 \text{ km/h} \end{aligned}$$

We then substitute the values of distance and time in the formula



Note it!

The gradient of a line in distance-time graph is the average speed travelled by a moving object.

After reaching the village the boy rested for 30 minutes (from 12:00 hours to 12:30 hours) and the graph, figure 4 above, is showing line horizontally as there was no speed applied and no distance covered.

From 12:00 hours to 12:30 hours the graph is parallel to the x-axis (time-axis) and therefore its gradient is 0. The speed during this period is also 0.

After 12:30 hours, the boy goes back to school and the graph here begins to descend.

Calculating the Speed of the Boy Returning to School

The speed of the boy in his returning journey back to school is given as

$$\text{Speed} = \frac{\text{distance travelled}}{\text{time taken}};$$

where time taken = 3 hours (= 15:30 – 12:30) hours, distance covered = 20 km

$$\text{Therefore, Gradient} = \frac{\text{vertical increase}}{\text{horizontal increase}}$$

$$\text{Gradient (or speed)} = \frac{\text{distance travelled}}{\text{time taken}}$$

$$\begin{aligned} \text{Speed} &= \frac{20 \text{ km}}{3 \text{ h}} \\ &= 6\frac{2}{3} \text{ Km/h} \end{aligned}$$

We have substituted the values of distance and time in the formula here.

Or 6.7 km/h

From our calculations you should note that the boy travelled at low speed when travelling back to school compared to when he was going to his village. This can be noticed by the steepness of the line. The gradient of the line from school to the village is steeper than from village to school. You can study the diagram again to see the differences. If you feel there is something that you have not properly understood, read the example again. This will make it easy for you to work through the next example and any later discussions or problem solving exercises about distance-time graphs.

Let us do another example to improve our understanding of distance-time graphs. This example is similar to the previous one. The only difference is that, this time we will draw the graphs to describe the journey for two people on the same Cartesian graph.

Example 2:

Suppose a girl also starts off cycling from the same school at the same time (10:00 hours) as the boy and she reached the same village after 3 hours. She rests for 30 minutes and starts off for school and reaches the school at 15 hours. We should answer the following questions together

- Draw on the same diagram the distance-time graph for both the boy and the girl's journey.
- What was the average speed for her journey from the village to school?
- At what time did she overtake the boy?
- After travelling for how many kilometres did she overtake the boy?

Solution

Step 1: We should draw the graph for the journey of the boy the way we did in Example 1.

Step 2: We should determine the time the girl will arrive at the village if it takes her 3 hours and departed to school at 10:00hours. Thus:

$$10:00 \text{ hours} + 3 \text{ hours} = 13:00 \text{ hours}$$

Step 3: We should draw the graph for her trip to the village on the Cartesian graph. We will do so by drawing a straight line from the point where the x-axis (time-axis) and y-axis (distance-axis) meet to the point where vertical line for 13:00 hours intersect the horizontal line for 20 km on the distance-axis.

Step 4: We will draw a horizontal line to represent the time she was at the village. During this time, there is no change in distance and the gradient is 0. This line is joined to the one we had drawn in step 3. That is, it is drawn from the vertical line for 13:00 hours to vertical line for 13:30 hours.

Step 5: We will finally draw the last line to represent her returning trip back to school. This straight line is drawn from where the last line ended to the point where the vertical line for 15:00 hours meets the x-axis (time-axis).

The completed graphs for both the boy and the girl will appear as shown below. You should note that the graph for the boy is in solid line while the graph for the girl is in broken line.

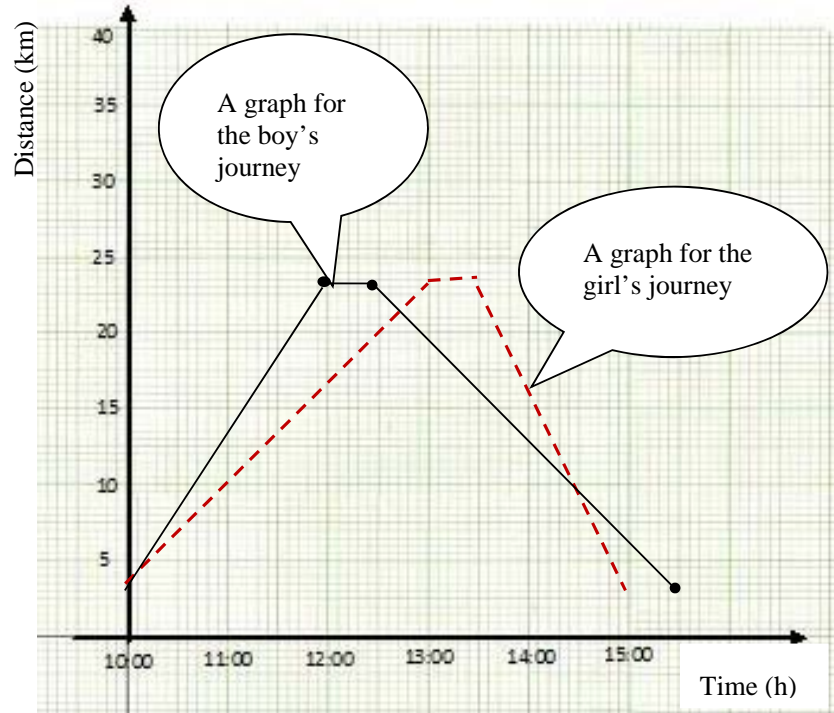


Figure 5: Distance-time graphs for the boy and the girl

Let us answer the next question.

Question: (b) what was the average speed for her journey from the village to school?

Solution

Her average speed is equivalent to the gradient of the straight line from the village to her school

Gradient = $\frac{\text{distance travelled}}{\text{time taken}}$; where distance = 20 km and

time = 1 hour 30 minutes

Therefore,

$$\begin{aligned} \text{Gradient} &= \frac{20 \text{ km}}{1 \text{ h } 30 \text{ min}} \\ &= \frac{20 \text{ km}}{1.5 \text{ h}} \\ &= 20 \div 1.5 \end{aligned}$$

We convert 1h 30min to hours. We express 30 min as $\frac{30}{60} = \frac{1}{2} = 0.5 \text{ h}$
Therefore, $1 \text{ h} + 0.5 \text{ h} = 1.5 \text{ h}$

$$= 20 \div 1\frac{1}{2}$$

We change the decimal fraction to mixed number

$$= 20 \div \frac{3}{2}$$

We change the mixed number to improper fraction

$$= 20 \times \frac{2}{3}$$

We multiply the multiplicand; that is 20 by the reciprocal of $\frac{3}{2}$, that is $\frac{2}{3}$

$$= \frac{40}{3}$$

$$= \mathbf{13.3\text{km/h}}$$

After doing Example 1, I am sure you were able to follow the above steps with ease. Let us do the next question together.

Question: (c) At what time did she overtake the boy?

Solution

To find the time when the girl overtook the boy, we take down the time where the two graphs intersect each other when the two cycled from the village to school. From the graph the two graphs intersect at the vertical line for 14:30 hours.

Let us move to the last question.

Question: (d) After travelling for how many kilometres did she over take the boy?

Solution

To answer this question we should take down the number for the horizontal line, representing the distance, that is passing through the point of intersection we earlier discussed in question (c).

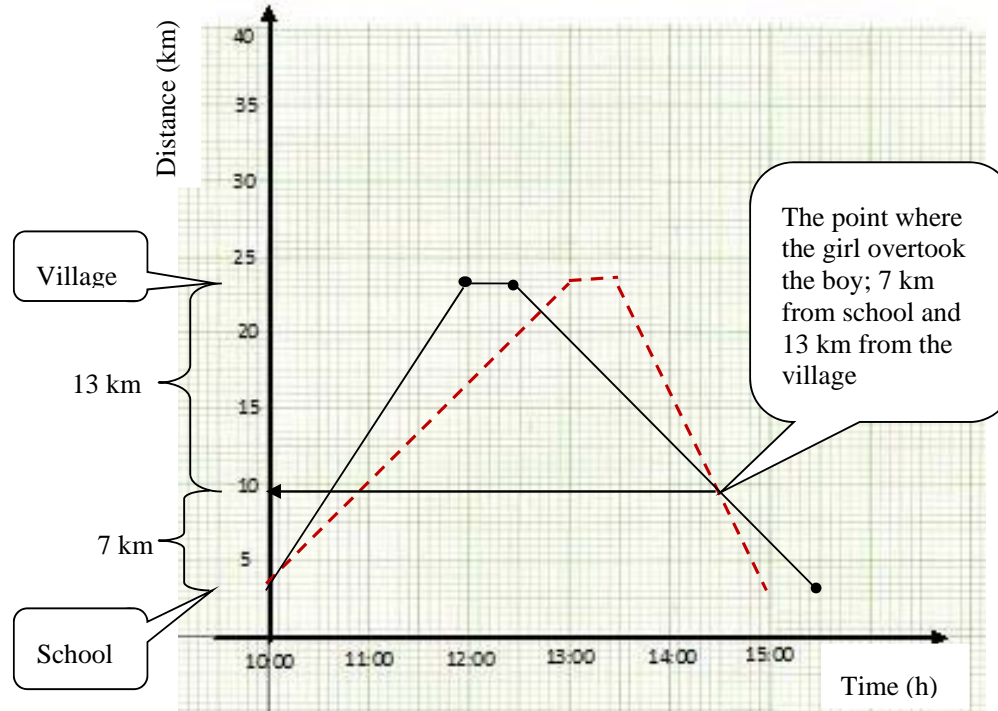


Figure 6: Diagram shows the point where the girl overtook the boy

Since the boy and girl were moving away from the village which is 20 km away from school, we subtract 7 km, the distance from school to the point where the girl overtook the boy, from 20 km.

Thus; $20\text{km} - 7\text{ km} = 13\text{ km}$

Therefore, the distance from the village where the girl overtook the boy is about **13 km**.

Check 13 km, the distance we have just calculated, from the distance-time graph above in Figure 6.

In the two examples we have just gone through, we drew distance-time graphs. You were taken through steps to draw the graphs. In the first example, we focused on drawing a single graph while in the second example we added one more graph on the same graph paper. We went further to answer questions based on the graphs.

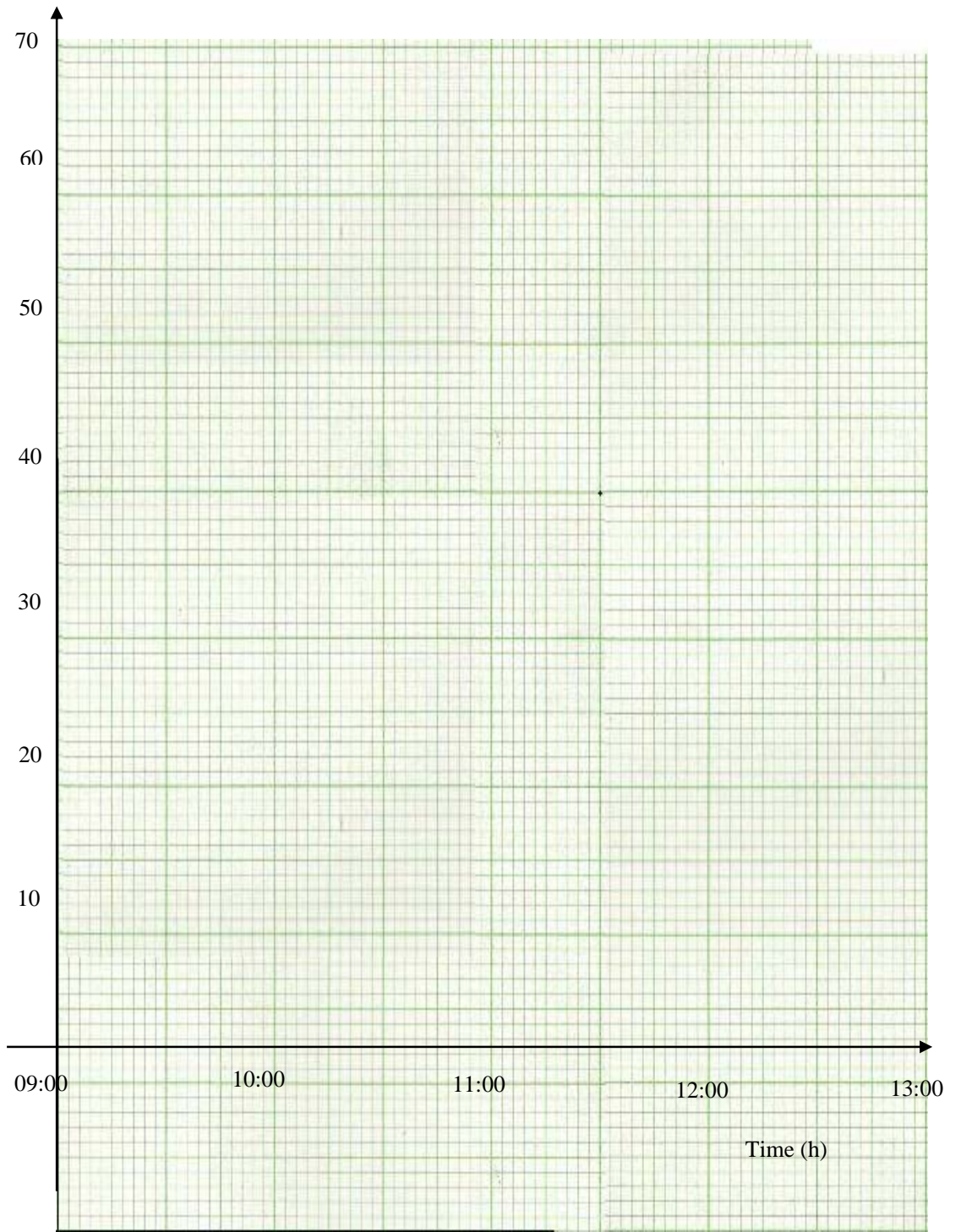
Now you should do the following activity. This activity is similar to example 1 since it will involve one graph. You will draw the distance-time graph on the graph paper provided and answer the questions based on the graph. This activity is important as it will help you understand how to draw the graph better.



Activity 2

Mr. Lungu leaves home at 09 00 and drives at a speed of 20 km/h. After $\frac{3}{4}$ hour he increases his speed to 45 km/h and continues at this speed until 10 45. He stops from 10 45 until 11 30 and then returns home at a speed of 50 km/h.

- (a) On the Cartesian graph below, draw a travel graph to illustrate the journey described above. The scale used to draw the axes is: 2 cm to 10 km on the distance-axis and 2 cm to 30 minutes on the time-axis.
- (b) Use the graph to find the approximate time at which she arrives home.





Feedback

Feedback

From the two examples we discussed earlier, you should have found it not too difficult to draw the required graph. As in the previous examples we should follow the following steps to draw a travel graph to illustrate Mr. Lungu's journey.

Step 1: *We have to calculate the distance travelled in $\frac{3}{4}$ hours at the speed of 20 km/h.*

Thus, distance = speed \times time; where speed = 20 km/h, time = $\frac{3}{4}$ hour

Therefore, Distance = 20 km/h \times $\frac{3}{4}$ h = 15 km

Step 2: *We should calculate the time he arrived at 15 km for $\frac{3}{4}$ hour taken. We can as well change $\frac{3}{4}$ hour to minutes. You should note that $\frac{3}{4}$ of an hour is 45 minutes. Therefore, we will add 45 minutes to 09:00 to get 09:45 hours. This means that Mr. Lungu arrived at the distance of 15 km at 09 45 hours.*

Step 3: *We plot the coordinate of the points where 09:45 on the time-axis intersect 15 km on distance-axis. Then we should draw the straight line from the point where 09: 00 on the time-axis intersect 0 km on the distance-axis to the coordinates of the point we earlier plotted*

Check the coordinates we have just discussed on the diagram below and the line joining them.

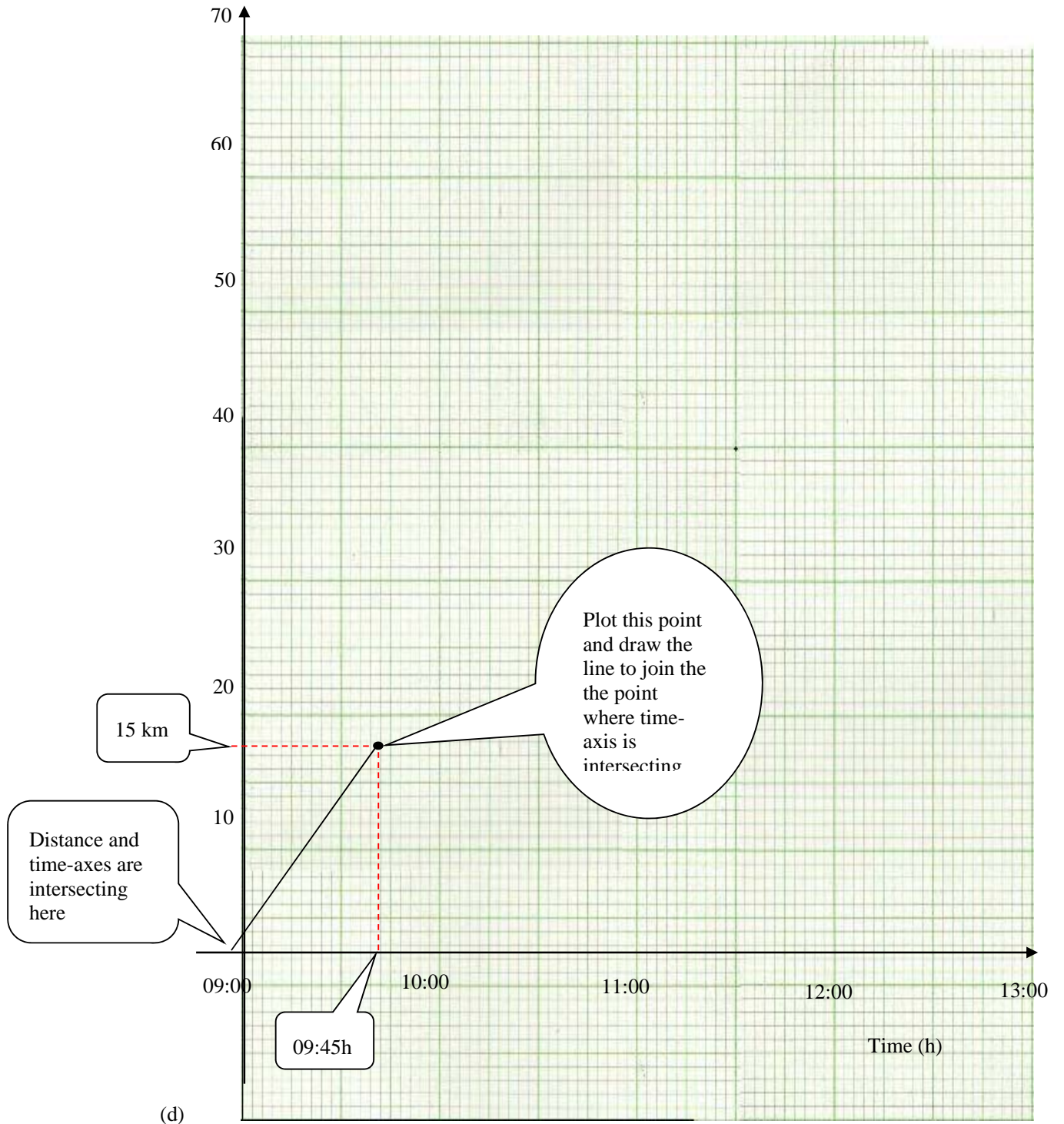


Figure 7: Distance-time graph

Let us move to the next step.

Step 4: We should note that the next speed Mr. Lungu travelled is 45 km/h. We should also note that Mr. Lungu travelled at this speed from 09:45 to 10:45. This gives us the difference of 1 hour. Therefore, when we consider the speed of 45 km/h we can conclude that Mr. Lungu took 1 hour to travel the distance of 45 km. We should therefore add 45 km to 15 km already travelled in the first 45 minutes to get 60 km.

Step 5: We should plot the coordinates of the point where 10:45 on the time-axis intersect 60 km on the distance-axis. Then we should extend the graph from where it ended to this newly plotted point.

Check the extension of the graph in the figure below.

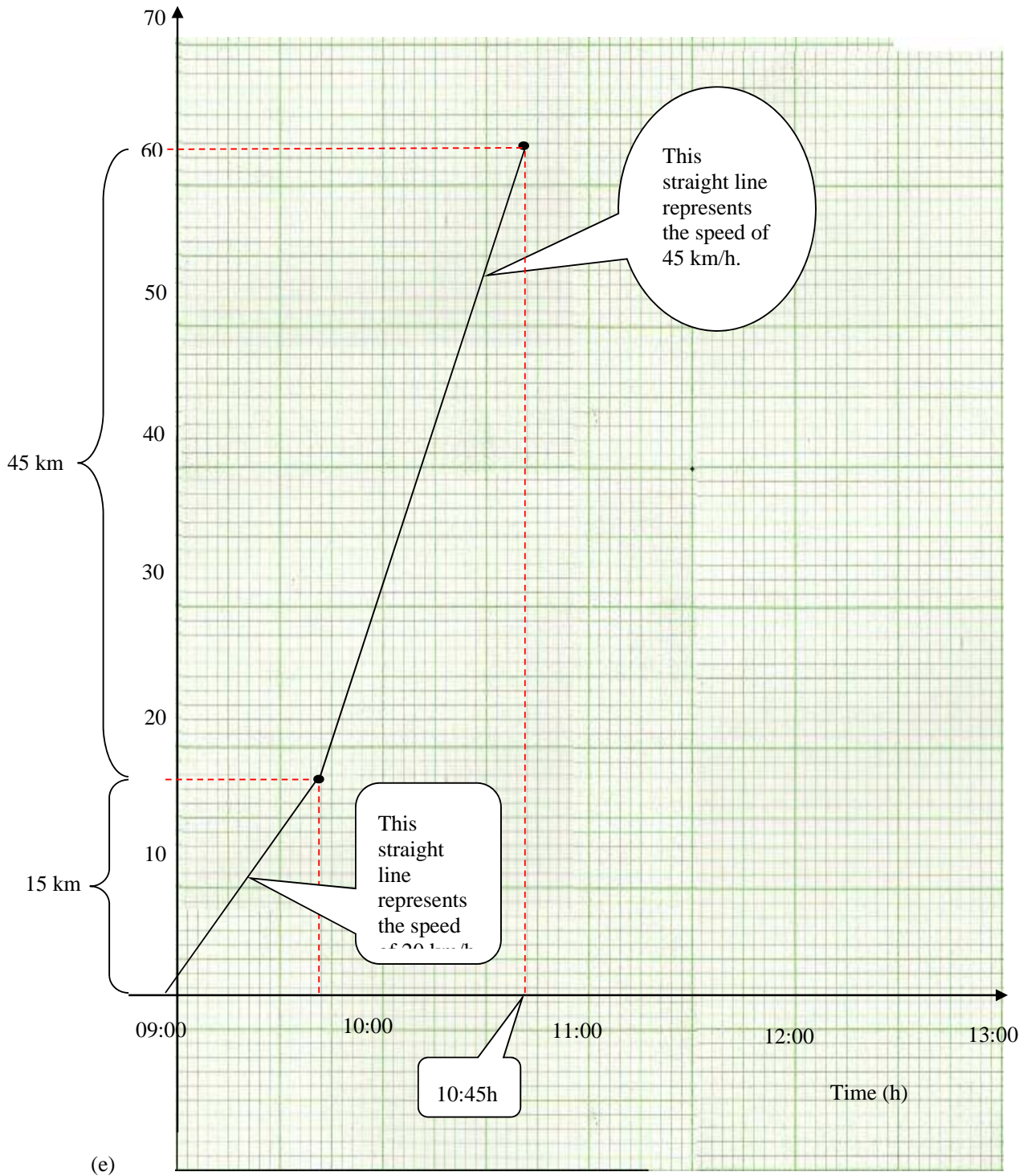


Figure 8: Distance-time graph for the distance of 60 km in 1 hour 45 minutes

Step 6: We should plot the time Mr. Lungu stopped. This time is from 10:45 to 11:30. Since there is no change in distance, the graph will be horizontal.

Check on the diagram in Figure 9 the time Mr. Lungu had stopped from 10:45 to 11:30.

Step 7: We should calculate the time Mr. Lungu took to return home. You should note that Mr. Lungu had travelled the distance of 60 km. You also note that he returns home at the speed of 50 km/h. Therefore, the time taken is

$$\mathbf{Time} = \frac{\mathbf{distance}}{\mathbf{timetaken}}$$

$$\mathbf{Time} = \frac{60\mathbf{km}}{50\mathbf{km/h}}$$

$$= 1\frac{1}{5} \text{ hour or}$$

$$= 1 \text{ h } 12 \text{ minutes}$$

Step 8: We should determine the time Mr. Lungu arrives home. To do this we have to add 1 hour 12 minutes to 11 30 to get 12 42.

Step 9: Finally we should draw the straight from where the graph ended to the point where 12 42 meet 0 km.

The completed travel graph describing Mr. Lungu's journey will appear as shown below.

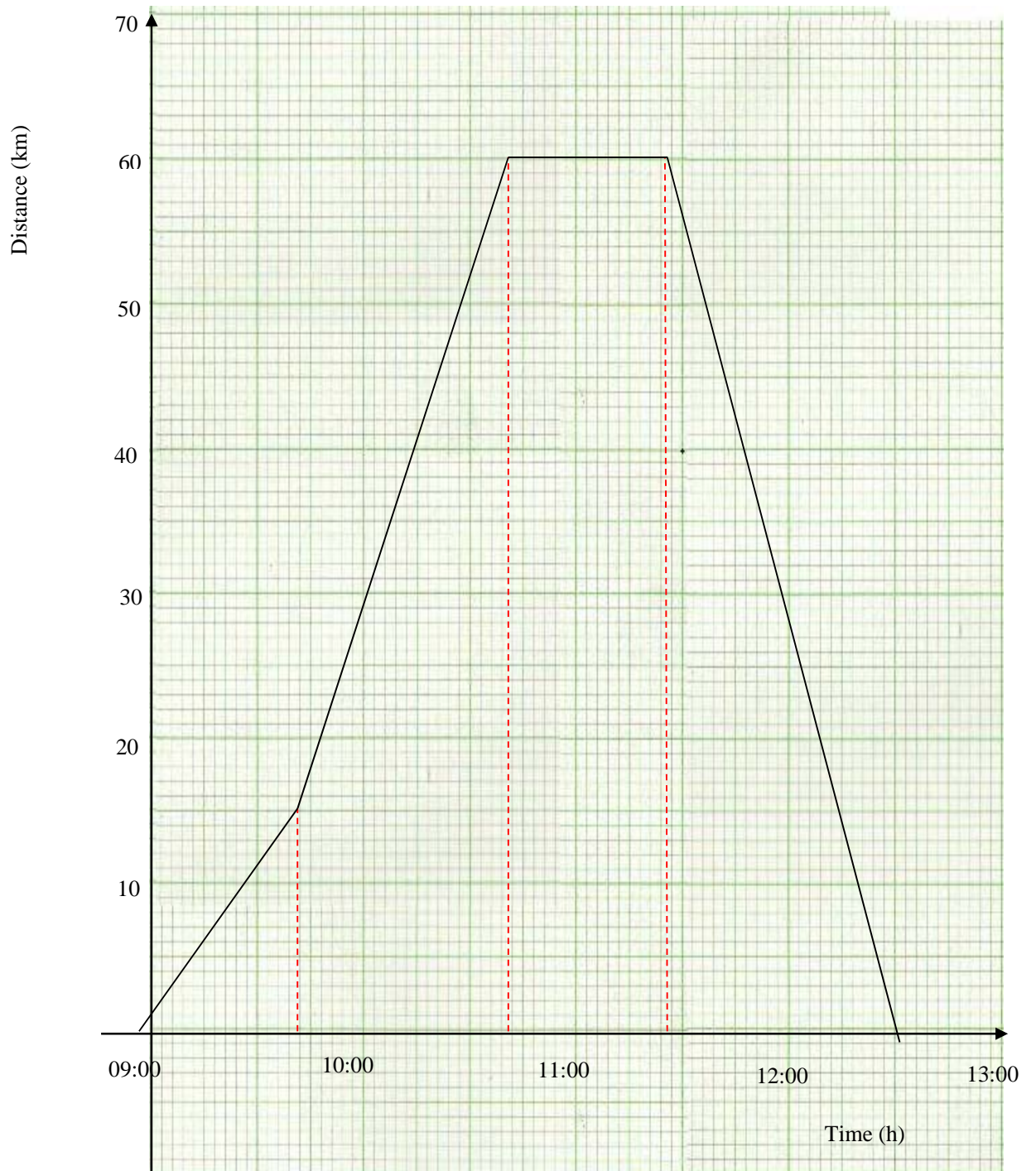


Figure 9: Diagram showing the graph of Mr.Lungu's journey

Let us do the last activity before you are given the chance to do topic exercise 1. The activity you are about to do is similar to the one you have just done. The only difference is that this time you will draw two graphs on the same diagrams. You will also answer questions based on the graphs. The activity is important as it will help you understand how to draw distance-time graphs.

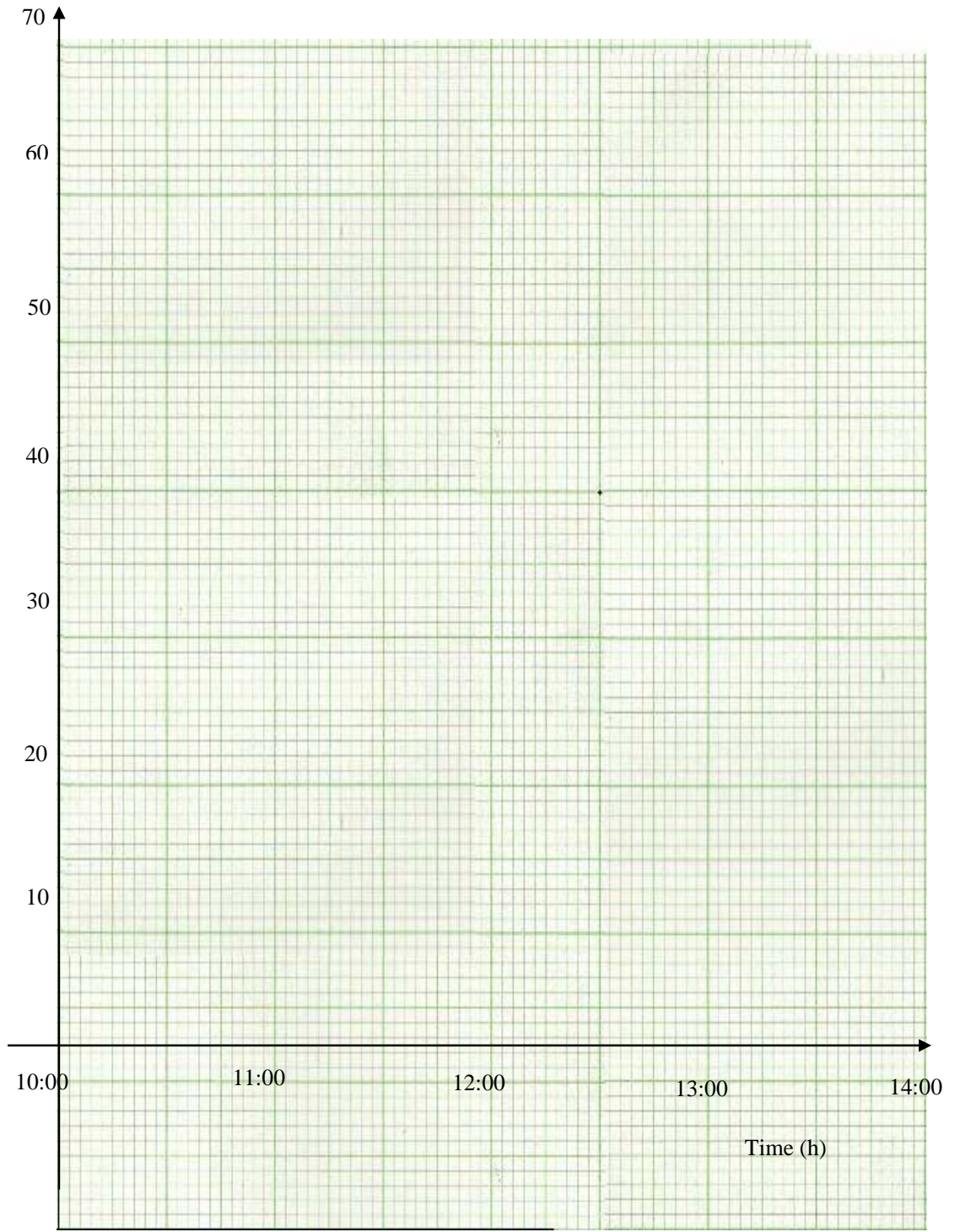
Activity 2: Drawing distance-time graph



Activity 3

At 10:00 Beatrice leaves home and cycles to her grandparents' house which is 70 km away. She cycles at a speed of 20 km/h until 11:15, at which she stops for $\frac{1}{2}$ hour. She then completes the journey at a speed of 30 km/h. At 11:45 Beatrice's brother, Isaac, leaves home and drives his car at 60 km/h. Isaac also goes to his grandparents' house and uses the same road as Beatrice.

- (a) Use the Cartesian graph below to draw a travel graph to illustrate Beatrice and Isaac's journeys as described above.
- (b) At approximately what time does Isaac overtake Beatrice?





Feedback

Feedback

To draw the travel graphs that describe the journeys for Beatrice and Isaac, we need to calculate the distance they travelled at the speed given.

Since Beatrice started her journey first we will start working out her graph first.

Details for Beatrice

Step 1: The distance travelled at 20 km/h from 10 00 to 11 15 is

$$\begin{aligned}
 \text{Distance} &= \text{speed} \times \text{time} \\
 &= 20 \text{ km} \times (11 \text{ 15} - 10 \text{ 00}) \text{ h} \\
 &= 20 \times 1 \text{ h } 15 \text{ min} \\
 &= 20 \times 1\frac{1}{4} \\
 &= 20 \times \frac{5}{4} \\
 &= 25 \text{ km}
 \end{aligned}$$

Step 2: After stopping for $\frac{1}{2}$ hour, Beatrice departed at 11 45 hours since $11 \text{ 15} + 00 \text{ 30} = 11 \text{ 45}$ hours. Then we should draw horizontal line from 11 15 to 11 45. During this time, there is no change in distance.

Step 3: Beatrice completed the journey at the speed 30 km/h. We should calculate the distance remaining as follows:

$$\text{Distance remaining} = 70 \text{ km} - 25 \text{ km} = 45 \text{ km}$$

$$\begin{aligned}
 \text{Time taken} &= \frac{\text{distance}}{\text{speed}}; \text{ where distance} = 45 \text{ km, speed} = 30 \text{ km/h} \\
 &= \frac{45 \text{ km}}{30 \text{ km/h}} \\
 &= 1 \text{ h } 30 \text{ min}
 \end{aligned}$$

Therefore, the time Beatrice arrived at home is worked out as follows:

$$\begin{aligned}
 \text{Time arrived} &= 11 \text{ 45} + 01 \text{ 30} \\
 &= 13 \text{ 15 hours}
 \end{aligned}$$

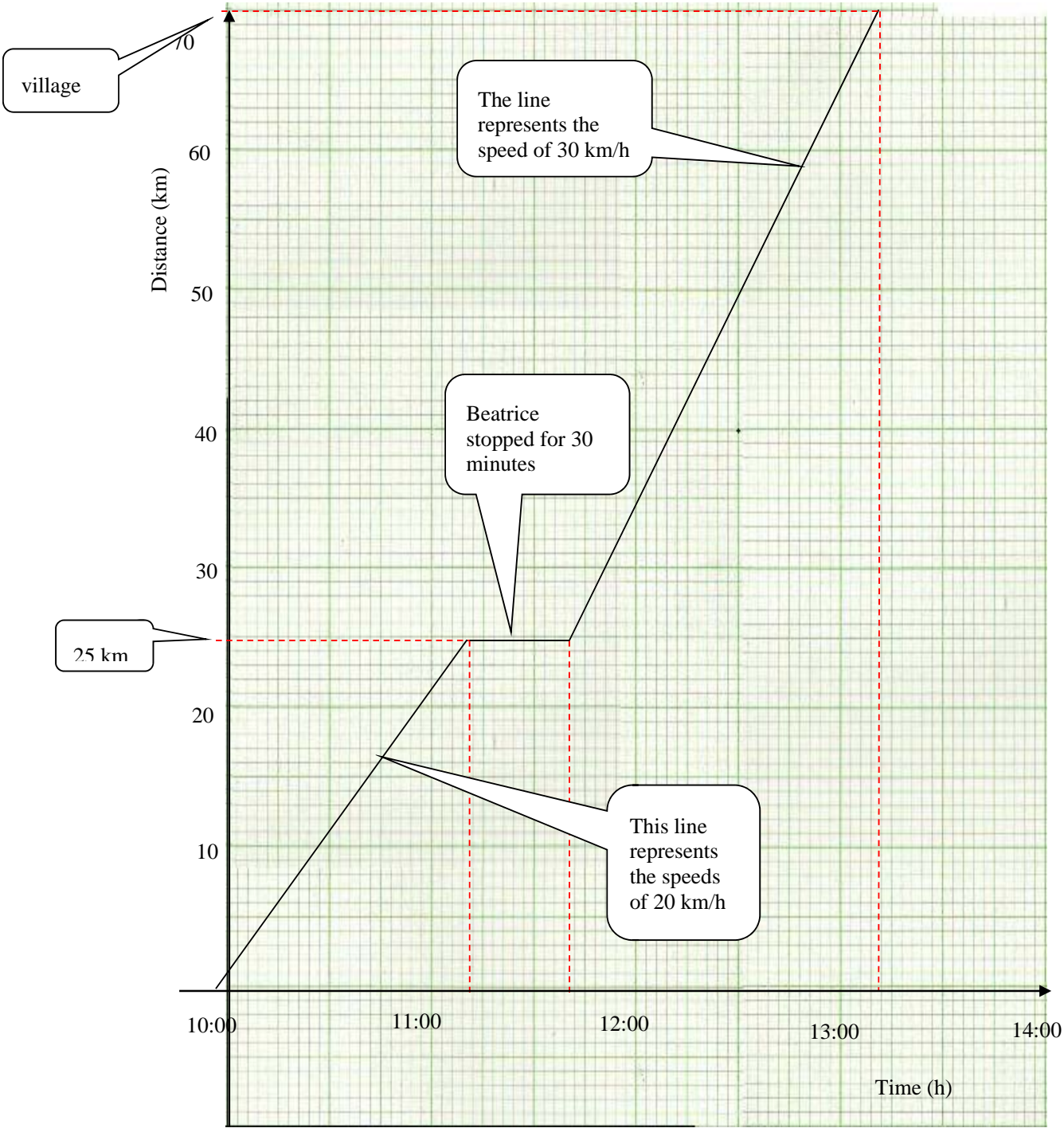


Figure 13: Diagram showing travel graph for Beatrice

Details for Isaac

Step 1: We should calculate time Isaac took for his journey.

$$\begin{aligned}\text{Therefore, time} &= \frac{\text{distance}}{\text{speed}}; \text{ where distance} = 70 \text{ km, speed} = 60 \text{ km/h} \\ &= \frac{70\text{km}}{60\text{km/h}} \\ &= 1\text{h } 10 \text{ min}\end{aligned}$$

Therefore, Isaac arrived at his grandparent's house at

$$11:45 + 01:10 = 12:55 \text{ hours}$$

Step 2: We should then draw the straight line from 11:45 on time-axis to the point where 12:55 on time-axis intersect 70 km on the distance-axis.

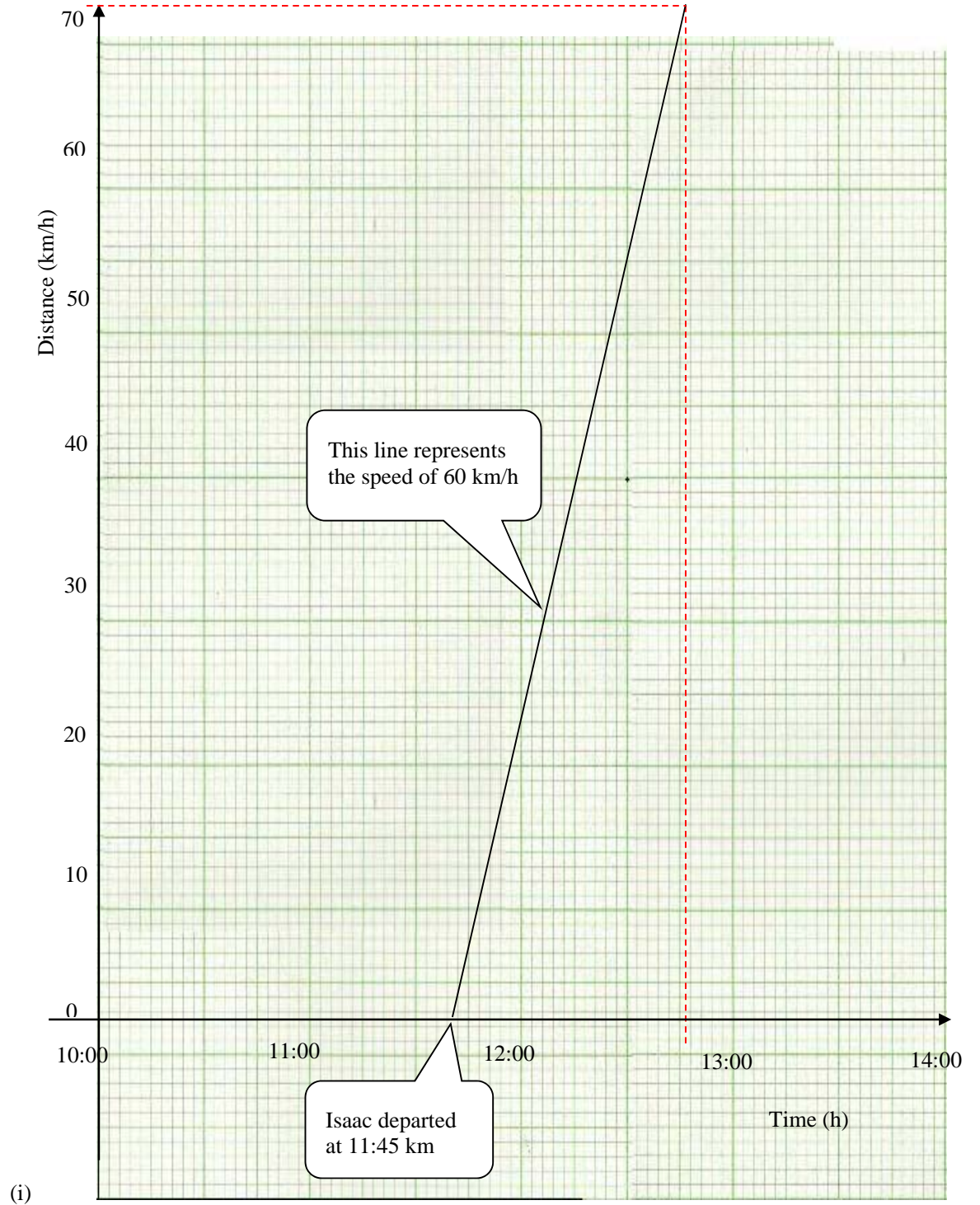
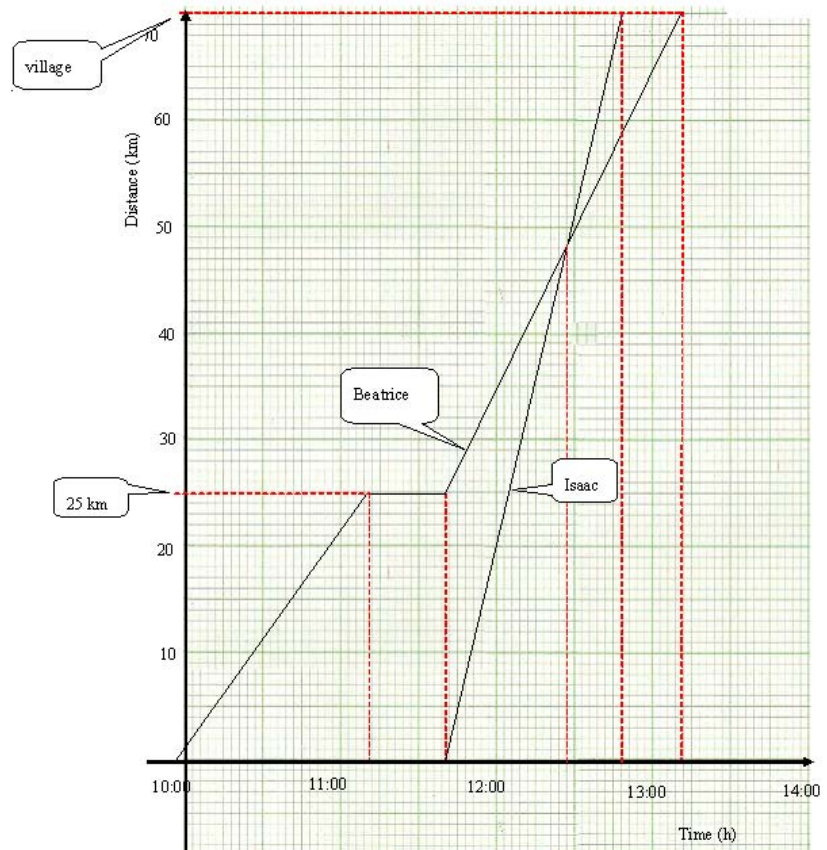


Figure 14: Diagram showing travel graph for Isaac

The completed travel graphs for Beatrice and Isaac's journeys is shown below.



(j) Isaac overtake Beatrice at approximately at 12 25

You have come to the end of topic 1. You should read the topic summary that follows.

Topic 1: Summary



Summary

In this first topic of unit 1: Distance-time graph, you have learned how to draw and interpret distance-time graphs. You have learned that the gradient of the graph distance-time graph represents the speed. The gradient of the straight line is given by

$$\text{Gradient} = \frac{\text{change in vertical-axis}}{\text{change in horizontal-axis}} \text{ or}$$

$$\text{Speed} = \frac{\text{change in distance-axis}}{\text{change in time-axis}}$$

You were also encouraged to work out the activities provided throughout the topic and compare your answers with the one provided in the feedback. The activities were intended to help you understand the content of the topic better. If you did not do very well in the activity, this means that you needed to go over the material again.

In this topic we discussed three quantities namely distance, time and speed. In distance-time graph, distance and time changed while speed ~~is~~ remained the same for any particular graph.

In the next topic we will discuss four quantities namely distance, time, speed and acceleration. This means that we will add one quantity to the three that we have discussed in this topic. In speed-time graph acceleration is constant.

Now you should do the following exercise to see how much you have learned in the whole topic. After completing the exercise, you should mark your own work by comparing your answers with those provided in the feedback soon after the exercise and then proceed to the next unit, only if you are satisfied with your progress.

Topic 2: Distance and Acceleration

This is the last topic in this unit. In this topic, you will learn how to calculate the distance covered by the moving object from the speed-time graphs. You will also learn how to find the acceleration from the speed-time graph. The same method of finding acceleration is similar to the one we used to find the speed from the distance-time graph in topic 1.

Like in topic 1 of this unit, there are some activities in this topic for you to do. These activities will help you understand the topic better. Like in the previous topics, there is topic exercise 2 at the end of the topic for you to do. After doing the exercise you should mark your own work by comparing your answers with the one provided in the feedback.

In this last topic of the unit, we will address the last unit outcome. That is:

- *Calculate* the distance covered in a specified time from a speed – time graph.

This outcome has been divided into objectives as listed below

Upon completion of this topic you will be able to:

- *Calculate* the acceleration and retardation from a speed-time graph.
- *Calculate* the distance, time and speed from a speed-time graph.



Objectives

The objectives you have just read through will help you to be aware of what you are expected to achieve after studying through this topic. Therefore, you are encouraged to work hard as before in order to achieve these objectives.

In topic 1: Distance-time graph, the calculations we dealt with had three quantities namely; **distance covered**, **time taken** and **speed**. We established that in distance-time graph, the time taken is represented by the horizontal axis while the distance covered is represented by the vertical axis and that the gradient of the graph represents speed. You should note that the quantities; distance and time in this graph are changing while the speed is constant.

In this topic; speed-time graph, our calculations will focus on four quantities namely, **distance covered, time taken, speed and acceleration.**

In speed-time graph, like in distance –time graph, the time taken is on the horizontal axis. Unlike in distance-time graph, the vertical axis in speed-time graph represents the speed.



Note it!

You should note that

- The speed and the time taken are changing quantities. The rising of the gradient of the straight line of the graph represents the **acceleration**. The term acceleration was defined in the terminology section as the rate of change of velocity.
- The increase in velocity is known as acceleration while the reduction of the speed is known as the **retardation** or **deceleration**.
- The area bounded by the graph and the time-axis represents the distance covered. This area is also known as the **area under the curve**.

Calculating Acceleration

We have already defined acceleration above as the rate of change of velocity. From the definition of acceleration, we can formulate the formula that we can use in solving the acceleration given the speed and the time. Thus:

$$\text{Acceleration} = \frac{\text{change in velocity}}{\text{time taken}}$$

Since in our study we are focussing on using the graph to calculate acceleration, then we will find acceleration by finding the gradient of the straight line of the graph. The straight rising of the graph from the left to the right is acceleration while the straight line of the graph sloping from the left to the right is the retardation.

Remember that we use the following formula to find the gradient of the straight line.

$$\text{Gradient} = \frac{\text{change in speed axis}}{\text{change in time axis}}$$



Note it!

The acceleration is equal to the gradient of a line in the velocity-time graph.

$$\therefore \text{the acceleration, } a = \frac{\text{change in velocity}}{\text{time taken}}$$

Before we do some example, let us look at some common units we used in speed-time graph.

Now you should note that there are some units we use when dealing with the speed-time graph. These units are discussed below.

Units Used

In the speed-time graphs, the commonly used units are

- **Metres per second** are used for **velocity**. This unit is written in short as **m/s**
- **Metres per square seconds** are used for **acceleration**. The unit is written in short as (m/s^2) .
- Since **retardation** and acceleration are the same though in the opposite direction, they carry the same type of the unit; metres per square seconds (m/s^2) . The retardation is also referred to as **deceleration**.
- We use **metres** (m) for distance.
- For the **time taken**, we use seconds (s).



Note it!

You should note that acceleration and deceleration are the same since both of them mean the rate of change of velocity. Deceleration is when the moving body is reducing the velocity and is said to have a negative acceleration or retardation.

$$\therefore \text{the deceleration, } -a = \frac{\text{change in velocity}}{\text{time taken}}$$

Let us look at the following example. This example will help you to understand how to find the acceleration and the distance covered in a given time and speed from the speed-time graph.

Example 1:

The diagram below is the speed-time graph of the first 30 minutes of a car journey. Calculate:

- Acceleration in the first 10 seconds.
- Calculate the distance covered in the first 30 seconds.

We work out this example together.

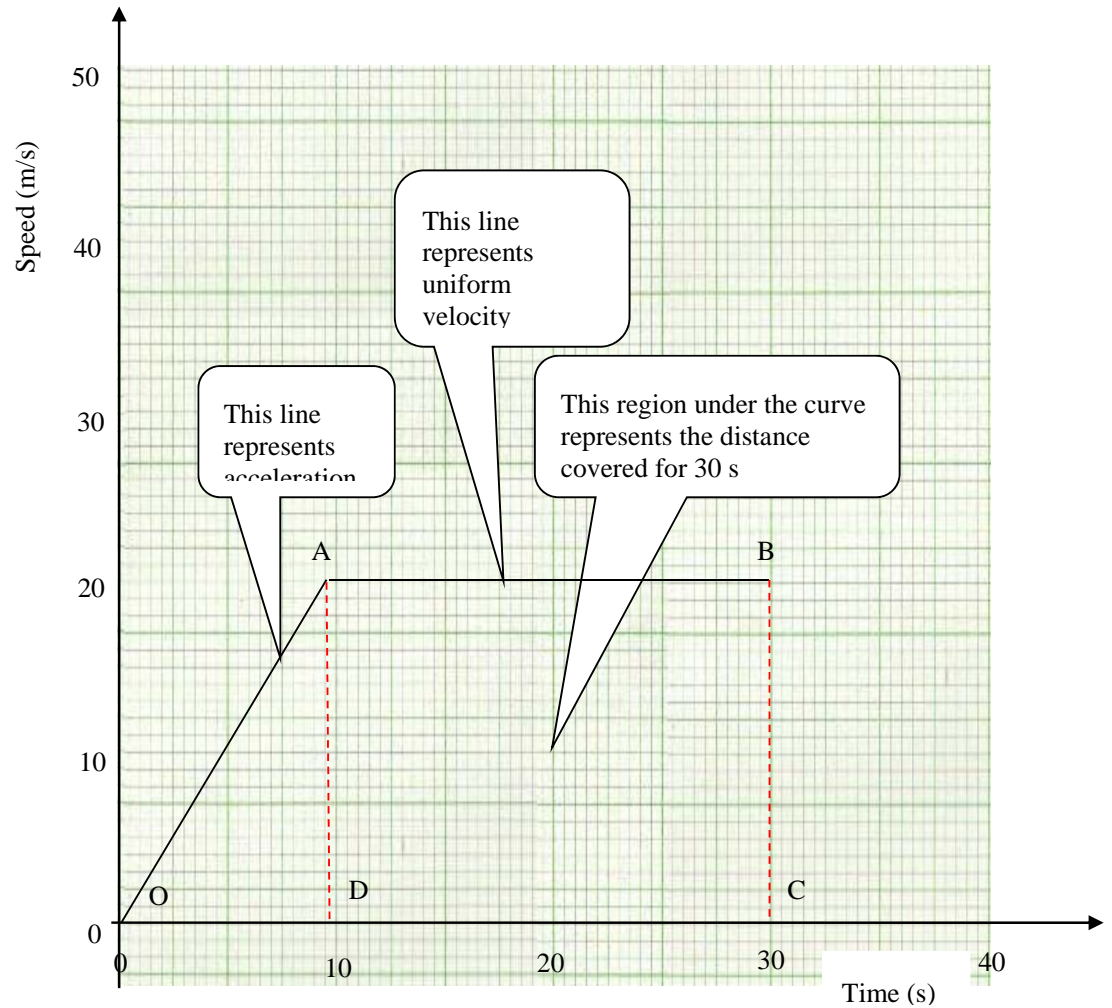


Figure 1: Speed-time graph

The diagram is the speed-time graph of the first 30 minutes of a car journey. Two quantities are obtained from such graphs. These quantities are acceleration and distance.

Solution

(a) Calculating Acceleration

We can find the acceleration for the distance of 20 m in 10 s as follows:

Acceleration = $\frac{\text{change in velocity}}{\text{time taken}}$; where distance = 20 km, time = 10 m/s

$$\text{Therefore, Acceleration} = \frac{20 \text{ m/s}}{10 \text{ s}}$$

$$\text{Acceleration} = \mathbf{2 \text{ m/s}^2}$$

(b) Calculating the distance from speed-time graph

Let us now calculate the distance covered in 30 seconds. You should note that the area of the curve in figure 1 is the shape of a trapezium ABCO. Trapezium is not new to you. In your junior certificate programme, you learned that a trapezium is a quadrilateral in which one pair of opposite sides is parallel but of different length.

The diagram below is an example of a trapezium. Note that side "a" and side "b" are opposite and parallel. The distance between the parallel sides is known as the height.

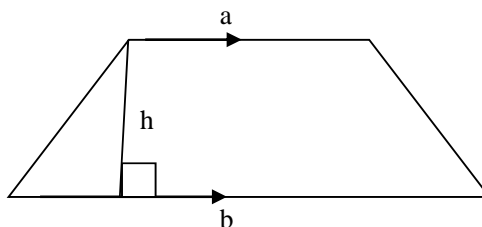


Figure 2: Trapezium

The formula we use to find the area of trapezium is:

Area = $\frac{1}{2} (a + b) h$; where 'a' and 'b' are parallel sides and 'h' is distance between the parallel lines.

Since distance travelled = area under graph, then

Distance = $\frac{1}{2} (AB + OD) \times AD$; where AB = 20 s, OC = 30 s and

$$AD = 20 \text{ m/s}$$

Therefore,

$$\text{Distance} = \frac{1}{2} (20 + 30) \text{ s} \times 20 \text{ m/s}$$

$$\text{Distance} = \frac{1}{2} \times 50 \text{ s} \times 20 \text{ m/s}$$

$$\text{Distance} = \mathbf{500\text{m}}$$

Note that the unit of **second** in 50 s and in 20 m/s will divide to give metre (m)

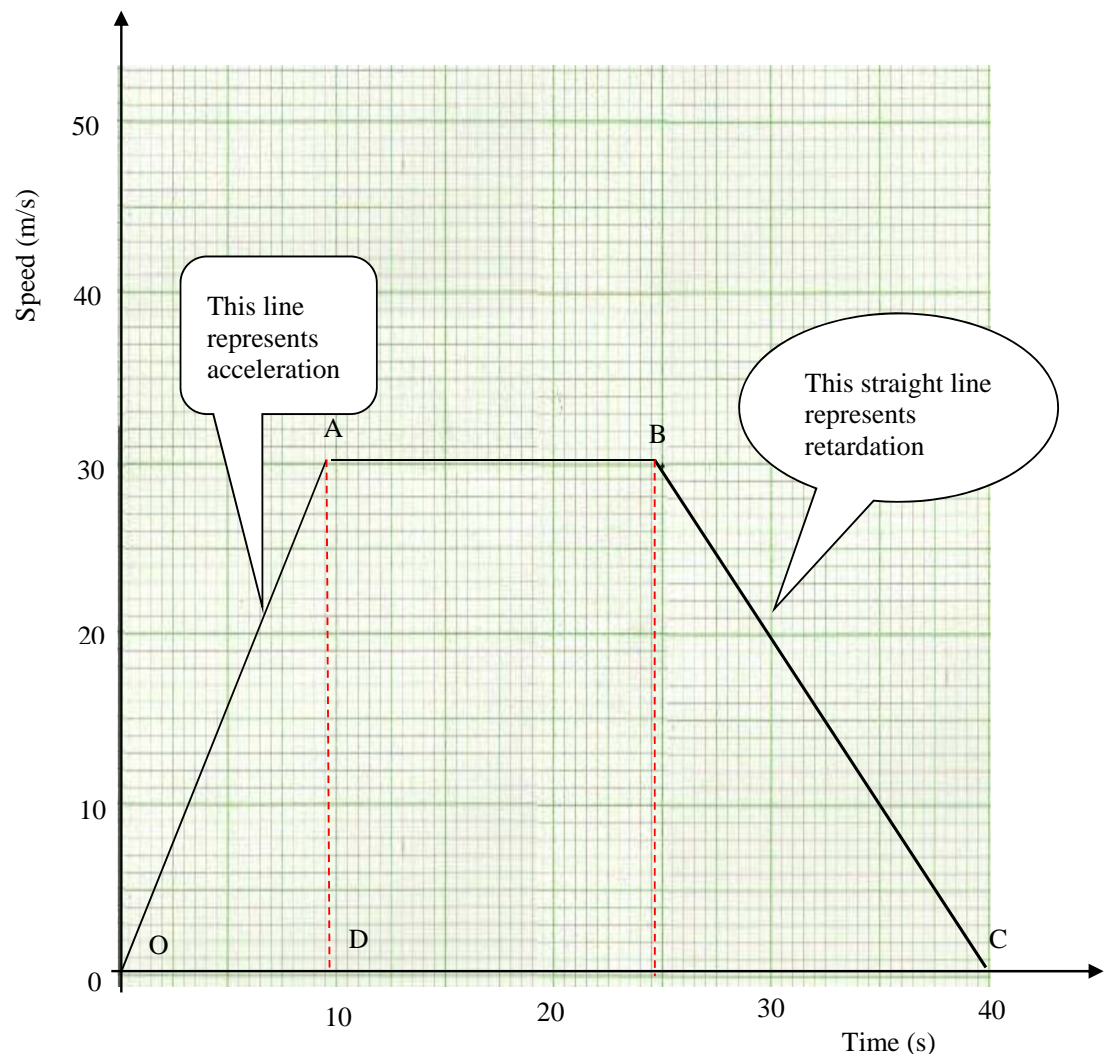
So far you have learned how to calculate the acceleration and the distance of the journey in a given time. Now you will do an activity in which you will calculate the average speed and retardation.



Activity 1

The diagram below is the speed-time graph of a car's journey. Use the diagram to calculate.

- the acceleration of the car
- the total distance taken by the car
- the average speed for the whole journey
- the retardation of the car.



Use the space provided below to write down your working and answers to these questions. If the space is not enough, you are free to use separate sheet of papers.



Feedback

Feedback

The graph shows the car's journey starting from a rest position (time is 'O', zero and speed is 'O', zero).

As the graph rises, the speed increases, so the car accelerates for 10 seconds and obtains a maximum speed of 30 m/s.

After 10 seconds, the car travels at a constant speed of 30 m/s for 30 seconds.

After 40 seconds, the car begins to retard or decelerate (Graph falling) for the next 20 seconds (60 s – 40 s).

After 60 seconds, the car then comes back to rest.

Once the graph is understood in this manner, then we can attempt to answer the questions.

Question (a): Acceleration = gradient of speed-time graph

$$\text{Acceleration} = \frac{\text{change in velocity}}{\text{time taken}}; \text{ where velocity} = 30 \text{ m/s,}$$

$$\text{Time} = 10\text{s}$$

$$\text{Acceleration} = \frac{30 \text{ m/s}}{10 \text{ s}}$$

$$\text{Acceleration} = 3 \text{ m/s}^2$$

Let us move to the next question.

Question (b) Calculate the total distance taken by the car

To find total distance taken by the car, we have to calculate the area of the curve bounded by the graph and the time-axis. The curve has the shape of the trapezium ABCO. Therefore,

Total distance = total area under the graph

$$\text{Total distance} = \frac{1}{2} (\text{AB} + \text{OC}) \times \text{AD}; \text{ where AB} = 30 \text{ m/s,}$$

$$\text{OC} = 60 \text{ s, AD} = 30 \text{ m/s}$$

$$= \frac{1}{2} \times (60 + 30) \text{ s} \times 30 \text{ m/s}$$

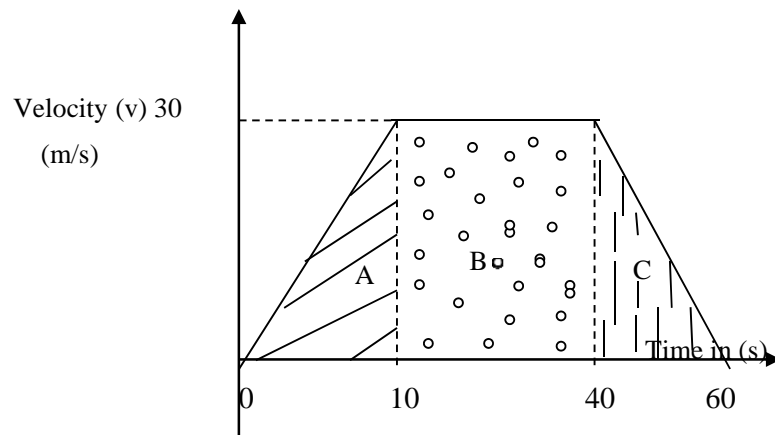
$$= \frac{1}{2} \times 90 \text{ s} \times 30 \text{ m/s}$$

$$= 45 \text{ s} \times 30 \text{ m/s}$$

$$= 1350 \text{ metres}$$

Therefore, the total distance taken by the car is **1350 metres**

Other than using the trapezium rule, we can also find the total distance by finding the areas of the three parts of the graph. The areas of the two triangles and the square in between can be expressed as:



Total dist = Area of triangle A + Area of rectangle B + Area of triangle C.

You should remember how to find the area of the triangle and rectangle because you learned this in your junior certificate programme. You learned that the formula we use to find the area of the triangle is

- Area of triangle = $\frac{1}{2} \times \text{base} \times \text{height}$ and
- Area of rectangle = length \times breadth

Therefore, the area of trapezium can as well be calculated as

Distance = area of Triangle A + Area of Rectangle B + area of Triangle C

$$\begin{aligned} \text{Distance} &= \left(\frac{1}{2} \times \text{base} \times \text{height}\right) + (\text{length} \times \text{breadth}) + \left(\frac{1}{2} \times \text{base} \times \text{height}\right) \\ &= \left(\frac{1}{2} \times 10 \times 30\right) + (30 \times 30) + \left(\frac{1}{2} \times 20 \times 30\right) \\ &= (10 \times 15) + 900 + (10 \times 30) \\ &= 150 + 900 + 300 \\ &= 450 + 900 \\ &= 1350\text{m} \end{aligned}$$

Therefore, the total distance covered is **1350 m**

Let us move to the next question

Question: (c) Calculate the average speed for the whole journey

In order to calculate the average speed for the whole journey, we have to use the formula for finding the speed. You used this formula in Topic 1 of this unit when you calculated the speed from the given distance-time graph.

Average speed = $\frac{\text{distance covered}}{\text{time taken}}$; where distance = 1350 m, time = 60 s.

$$\text{Average speed} = \frac{1350 \text{ m}}{60 \text{ s}}$$

$$= \frac{135}{6} \text{ m/s}$$

$$= 22\frac{1}{2} \text{ m/s or}$$

$$= \mathbf{22.5 \text{ m/s}}$$

Reduce the fraction by dividing both numerator and denominator by 10.

You can express the mixed number as a decimal number

Therefore, the average speed for the whole journey is 22.5 m/s

Let us move to the last question.

Question: (d) Calculate the retardation of the car.

You should note that, on the graph, the retardation is represented by the line sloping downward after 40 seconds until it touches the time-axis for the next 20 seconds at 60 seconds.

Gradient = $\frac{\text{change in distance}}{\text{time taken}}$; where distance = 30 m, time = 20 s.

$$= \frac{30 \text{ m}}{20 \text{ s}}$$

$$= \mathbf{1.5 \text{ m/s}^2}$$

Therefore, the retardation of the car is 1.5 m/s^2

You should do another activity. In this activity you will calculate acceleration and distance from the given graph. This activity is similar to the previous one. The speed-time graph may not appear as a graph under what is being discussed but it is.



Activity 2

Calculating the acceleration and the distance covered

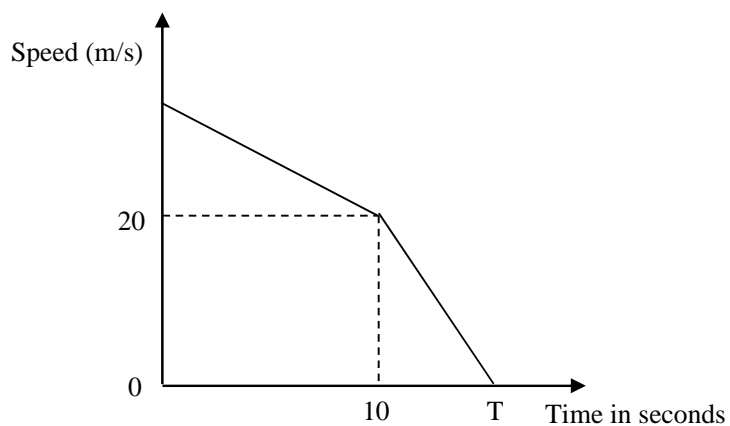


Figure 2:

The diagram shows the speed-time graph of a train's journey. The train decelerates uniformly for 10 seconds from a speed of V m/s to a speed of 20m/s. The train then decelerates further at the rate of 0.5m/s^2 .

Calculate

- The value of V if the distance travelled in the first 10 seconds is 400m.
- The retardation during the first 5 seconds
- The value of T .

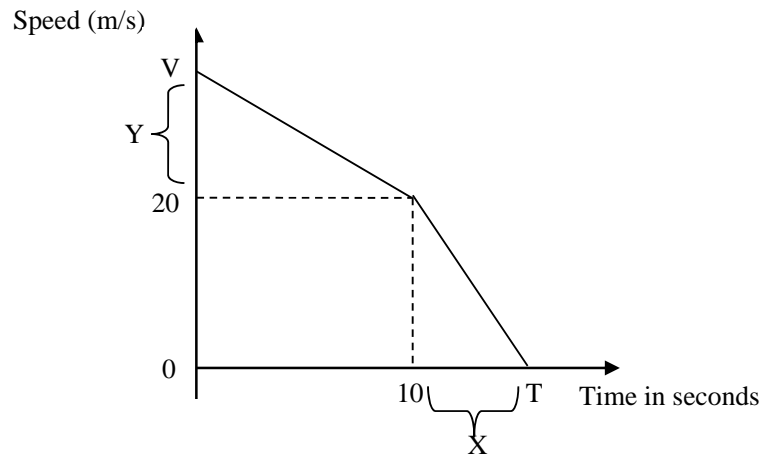
Write down your answer and work in the space below. You are free to use separate paper if you find the space provided is not enough.

Feedback



Feedback

- (a) *Distance travelled = area under the graph.
Let the speed between V m/s and 20 m/s be y .*



*Distance = $\frac{1}{2}(a + b) \times h$; where 'a' is the length in y -axis = 20
and $b = (20 + y)$ and $h = 10$ and distance = 400 m*

$$400 = \frac{1}{2}(20 + 20 + y) \times 10$$

$$400 = \frac{1}{2}(40 + y) \times 10$$

$$400 = 5(40 + y)$$

$$400 = 200 + 5y$$

$$200 + 5y = 400$$

$$5y = 400 - 200$$

$$\frac{5y}{5} = \frac{200}{5}$$

$$y = 40$$

$$\therefore V = 20 + 40$$

$$V = 60 \text{ m/s}$$

Therefore, the value of V is 60 m/s

To solve the value of 'y' with less difficulty, we rearrange the statement so that the values with 'y' are written first

Let us answer the next question

Question: (b) Calculate the retardation during the first 5 seconds .

You should note that, during the first 10 seconds the retardation is uniform. Therefore, we will find the gradient of the line in the first 10 seconds. You should note that the retardation for the first 5 minutes is the same retardation for the first 10 minutes. Therefore,

$$\text{Retardation} = \frac{\text{change in velocity}}{\text{change in time}}$$

$$\begin{aligned} \text{So retardation} &= \frac{60 - 20}{10} \\ &= \frac{40}{10} \\ &= 4\text{m/s}^2 \end{aligned}$$

Therefore, the retardation during the first 5 seconds is 4 m/s^2

Let us do the last question.

Question: (c) Calculate the value of T.

(c) Let the time between T and 10 seconds be x.

$$\text{Retardation} = \frac{\text{change in velocity}}{\text{change in time}}; \text{ given retardation} = 0.5\text{ m/s}^2$$

$$0.5\text{ m/s}^2 = \frac{20\text{ m/s}}{x}$$

$$0.5x = 20$$

$$x = \frac{20}{0.5}$$

$$x = \frac{20 \times 10}{0.5 \times 10}$$

$$x = \frac{200}{5}$$

$$x = 40$$

$$\therefore T = 10 + 40$$

$$= 50\text{s}$$

Since the denominator is a decimal number, we will multiply both the numerator and the denominator by 10. It is easier to divide by whole numbers.

Therefore, value of T is **50 seconds**

You have come to the end of topic 2. You should read the topic summary that follows.

Topic 2 Summary



Summary

In this second topic, you have learned how to calculate the acceleration and the distance from the speed-time graph. We calculate the acceleration of the moving object by finding the gradient of the sloping line of the graph. Acceleration is defined as the rate of change of velocity. Retardation is used to describe the decrease in velocity. The formula we use to find the acceleration or retardation is

$$\text{Acceleration (or retardation)} = \frac{\text{change in velocity}}{\text{time taken}}$$

Now you should do topic exercise 2. The exercise is intended to help you assess how well you understood the content of the topic. We encourage you to answer all the questions in the topic exercise. After completing the exercise, you should mark your own work by comparing your answers and working with those provided in the feedback that follows.

You should read the unit summary that follows.

Unit 7: Summary



Summary

In this unit there were two topics. In topic 1 you learned to draw and interpret distance-time and speed-time graphs. You learned how to find the speed of a moving object from the distance-time graph. The speed is determined by finding the gradient of the sloping straight line of the graph. The formula for finding the speed is

$$\text{Speed} = \frac{\text{change in distance}}{\text{change in time taken}}$$

In topic 2, you learned how to calculate acceleration and retardation from the speed-time graph. Acceleration is calculated by finding the gradient of the sloping of the straight line of the graph. Acceleration is given as

$$\text{Acceleration} = \frac{\text{change in velocity}}{\text{change in time taken}}$$

In topic 2, you also learned how to calculate the distance travelled in a given time from the speed-time graph. You learned that the area under the curve of the graph is equivalent to the distance covered.

By now you would have also completed the topic exercise 2. This means that, besides the activities that you did within the topics, you would have assessed your progress at the end of each of the two topics.

Congratulations on completing the seventh unit of the mathematics 11 course. There is no tutor-marked assignment in this unit. We trust that the activities and topic exercises have adequately helped you to assess your own progress. You will be asked to complete and submit a tutor-marked assignment at the end of unit 9.

The next unit, unit 8, is on symmetry in two dimensions. You will learn how to identify line of symmetry and rotational symmetry in a two dimensional shapes. You will also learn how to determine the order of rotational symmetry in dimensional shapes.

As already mentioned, there is no tutor-marked assignment in the unit. Tutor marked assignment will return in unit 9. If you have indeed completed all the end of topic exercises and are happy with the progress you have made in this unit, then you can get ready for the next unit.

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Topic Exercise 1



Assignment

Answer the following questions on a separate answer sheet.

1. A minibus and a taxi start off at the same time from Luanshya Town Centre station to Ibenga Girls High School a distance of about 25Km.

After travelling for 8 minutes the minibus reaches the first station at the council. The distance from Luanshya to the council station is 15 km. It stays at the station for 5 minutes and then starts off and travels for 5 km to reach station 2 at Boma in 2 minutes. Here it stays for 3 minutes and then travels another 5 km to reach Ibenga Girls High School.

The taxi goes straight to Ibenga High School and takes 20 minutes.

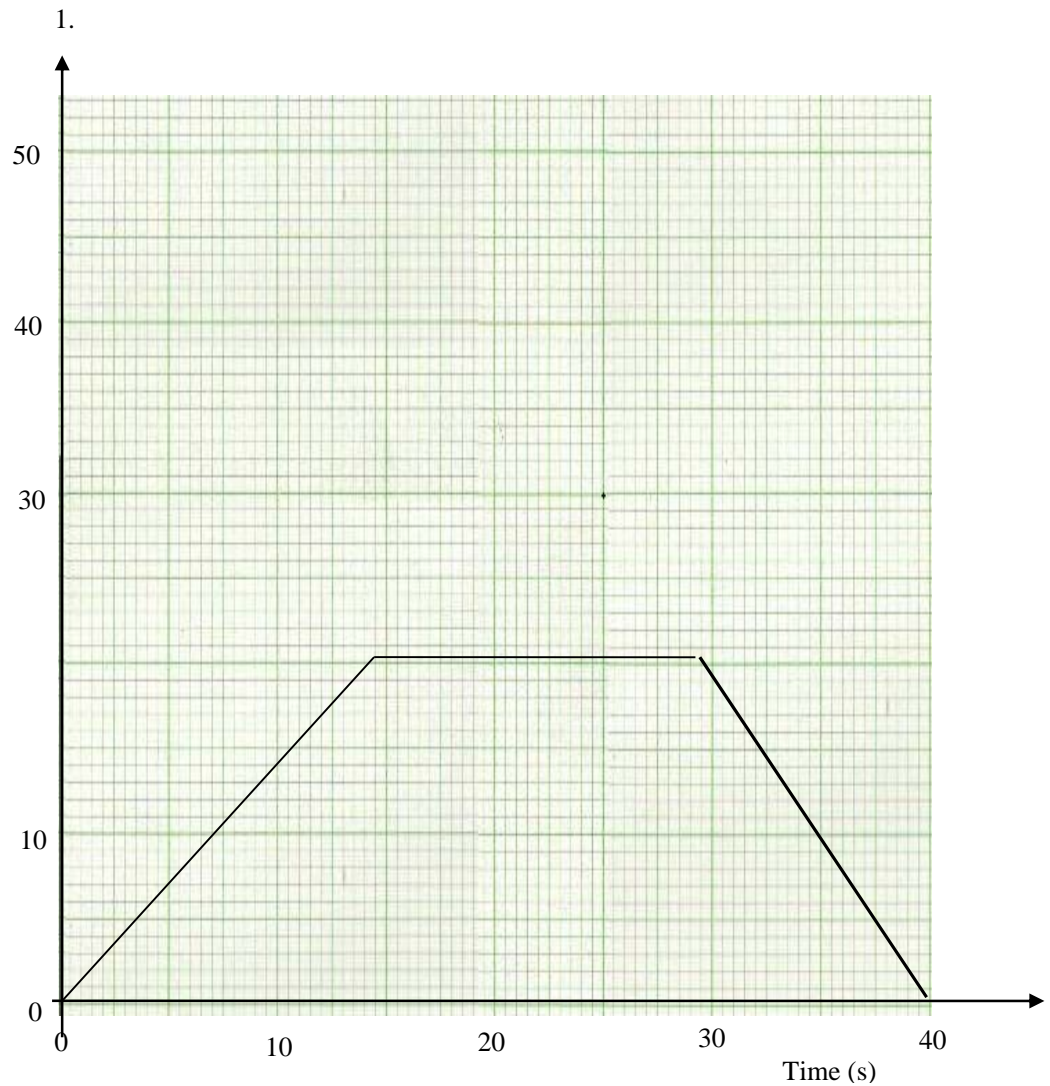
 - a) Taking a scale of 2cm to 5 minutes on the time axis and 2cm to 5 km on the distance axis, draw distance-time graphs for the minibus and taxi
 - b) At what average speed did the minibus travel up to the first station – council?
 - c) At what average speed did the taxi travel?
 - d) After how many minutes did the taxi overtake the minibus
 - (i) At Council Station
 - (ii) At Boma Station
2. Mrs. Banda leaves home at 08:00 and drives at a speed of 50 km/h. After $\frac{1}{2}$ hour she reduces her speed to 40 km/h and continues at this speed until 09:30. She stops from 09:00 until 10:00 and then returns home at a speed of 60 km/h.
 - (a) Draw a travel graph to illustrate the journey described, taking a scale of 2 cm to 10 km on the distance-axis and 2 cm to 30 minutes on the time-axis.
 - (b) Use the graph to find the approximate time at which she arrives home.

Topic Exercise 2



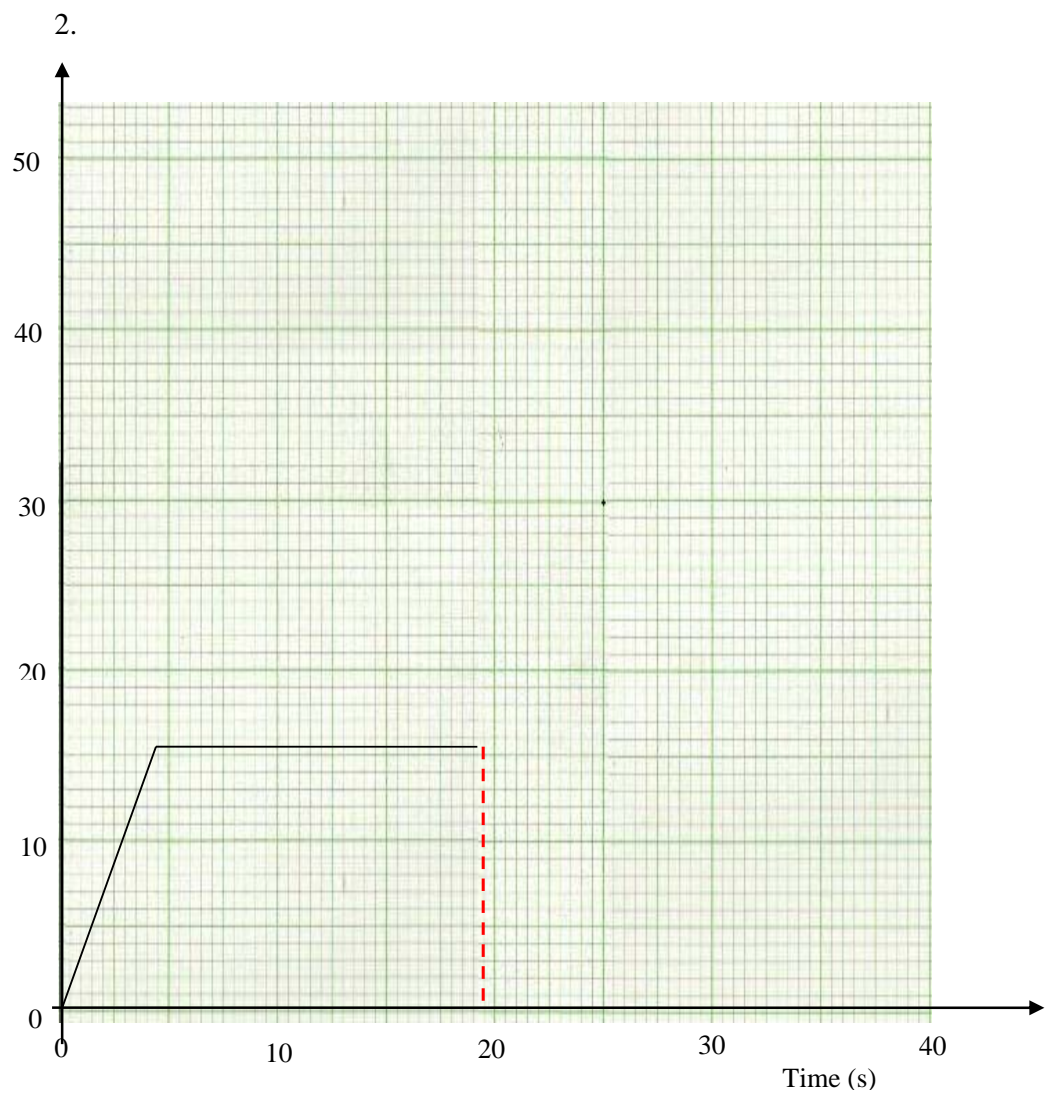
Assignment

Answer the following questions on a separate answer sheet.



Use the speed - time graph above for a cyclist's journey to calculate

- The acceleration
- The distance covered in the first 40 seconds
- The average speed for the whole journey.



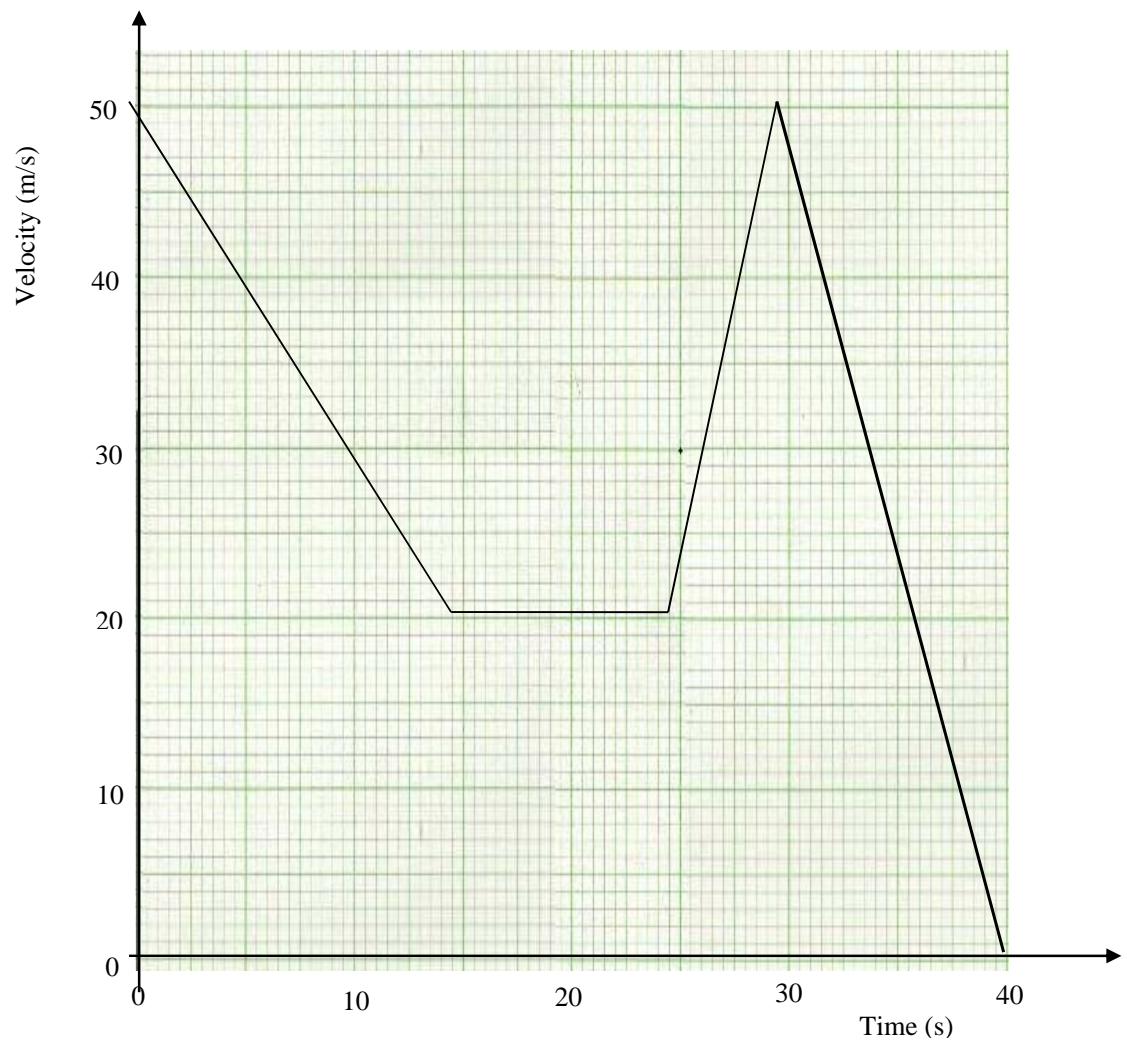
A horse rider's journey is shown in the speed time graph above. The rider accelerates for 5 seconds up to a speed of 15m/s. The horse maintains the same speed for the next 15 seconds.

Calculate

- the acceleration
- the distance covered in the first 15 seconds
- the acceleration when $t = 10$
- the total time taken to come to rest if the retardation after 20 seconds is $\frac{1}{3} \text{ m/s}^2$

- (e) The total distance covered before coming to rest.

3.



The diagram shows the speed-time graph of a racing motorist in a rally championship. The motorist decelerates uniformly from 50 m/s to 20m/s in seconds. Then he maintains the same speed for the next 15 seconds, after which he accelerates until he reaches the speed of 50m/s. He then decelerates again and comes to rest after 10 seconds.

Calculate:

- (a) The initial retardation.
- (b) The acceleration.
- (c) The total time taken for the whole journey of the total distance covered in 1 km.

Answers to Topic Exercise 1



Answers to Assignment 1

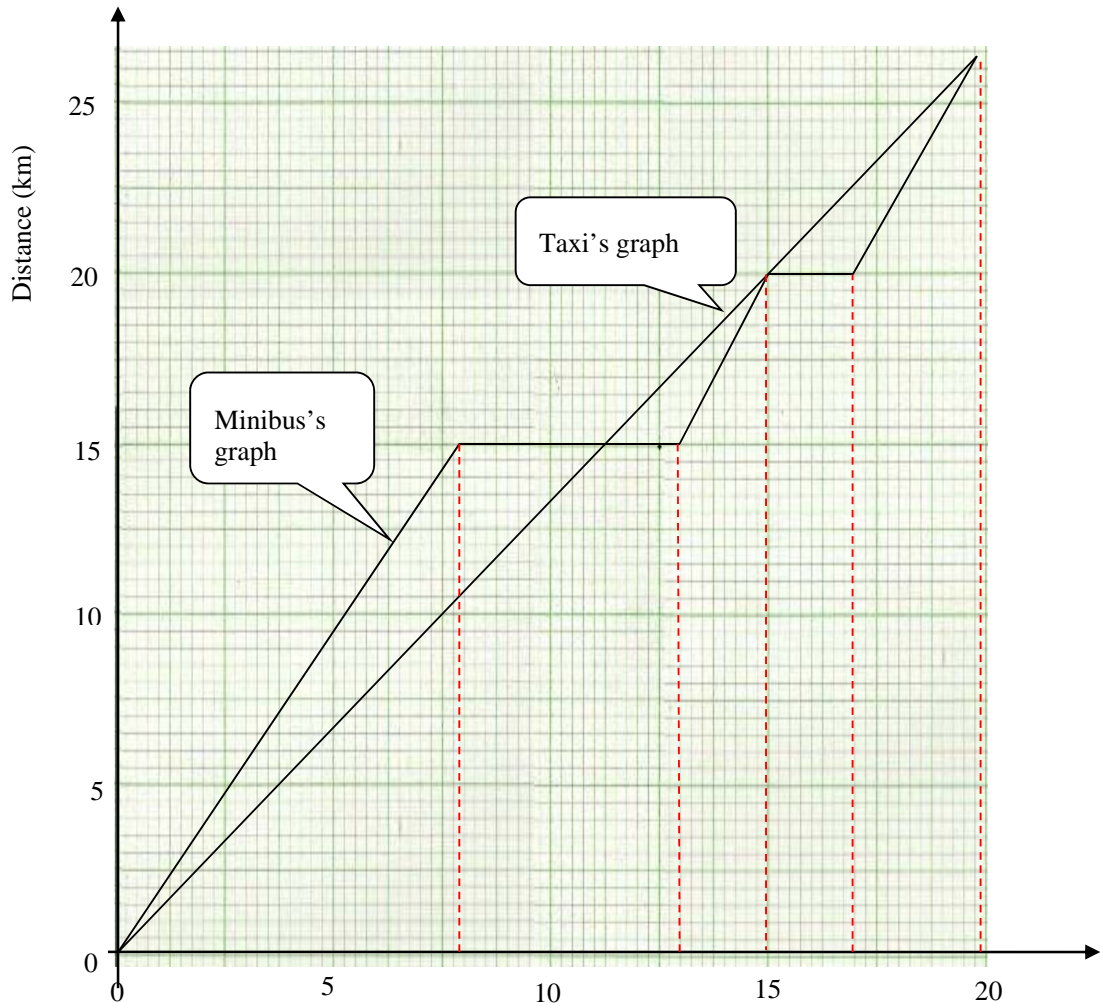
Before you check the answers in the feedback below you need to complete the exercise first. If you have completed the activity then you can now compare your answers with those provided in the feedback below.

1. (a) Distance-time graphs for the minibus and taxi

Scale: 2 cm: 5min or

20 mm:5min or

1 mm : 0.25 min



Time (min)

(b) The average speed of the minibus is equivalent to the gradient of the line

$$\text{Speed} = \frac{\text{change in distance}}{\text{change in time}}; \text{ where change in distance} = 15 \text{ km,}$$

$$\text{change in time} = 10 \text{ min.}$$

$$\begin{aligned} \text{Average speed} &= \frac{15 \text{ km}}{10 \text{ min}} \\ &= 1.5 \text{ km/min} \end{aligned}$$

Therefore, the average speed of the minibus upto the first station – council is 1.5 km/min

$$\text{(c) Speed} = \frac{\text{change in distance}}{\text{change in time}}; \text{ where change in distance} = 25 \text{ km,}$$

$$\text{change in time} = 25 \text{ min.}$$

$$\begin{aligned} \text{Average speed} &= \frac{25 \text{ km}}{25 \text{ min}} \\ &= 1 \text{ km/min} \end{aligned}$$

Therefore, average speed of the taxi is 1 km/min

(d) (i) The taxi overtook the minibus at council station after 15 minutes

(ii) The taxi overtook the minibus at Boma station after 20 minutes

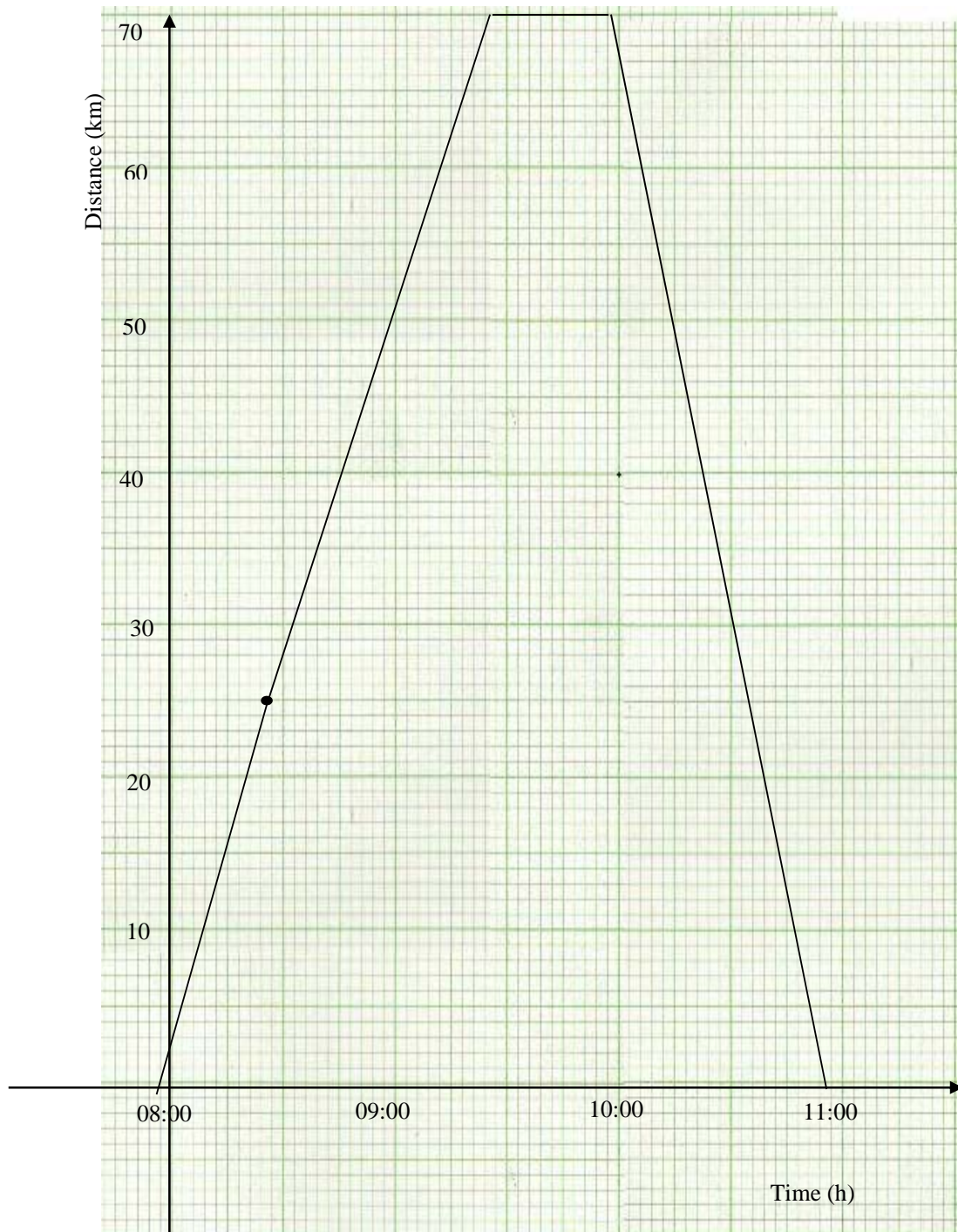
(a) Distance-time graphs for Mrs. Banda's journey

Let us move to the next step.

Step 4: We should note that the next speed Mr. Lungu travelled is 45 km/h. We should also note that Mr. Lungu travelled at this speed from 09:45 to 10:45. This gives us the difference of 1 hour. Therefore, when we consider the speed of 45 km/h we can conclude that Mr. Lungu took 1 hour to travel the distance of 45 km. We should therefore add 45 km to 15 km already travelled in the first 45 minutes to get 60 km.

Step 5: We should plot the coordinates of the point where 10:45 on the time-axis intersect 60 km on the distance-axis. Then we should extend the graph from where it ended to this newly plotted point.

Check the extension of the graph in the figure below.



Firstly, we should work out the distance covered for $\frac{1}{2}$ hour.

$$\begin{aligned}\text{Distance} &= \text{speed} \times \text{time} \\ &= 50 \text{ km/h} \times \frac{1}{2} \text{ h} \\ &= 25 \text{ km}\end{aligned}$$

Secondly, we should work out the distance covered for 1 hour (09 30h – 08 30h = 1 h)

$$\begin{aligned}\text{Distance} &= \text{speed} \times \text{time} \\ &= 40 \text{ km/h} \times 1 \text{ h} \\ &= 40 \text{ km}\end{aligned}$$

Note that the unit of **hour** in 40 km/h and in 1 h will divide to give kilometre (km)

We need to determine the time taken to reach home at the speed of 60 km/h, where the distance is 65 km.

$$\begin{aligned}\text{Time} &= \frac{\text{distance}}{\text{time}} \\ \text{Time} &= \frac{65\text{km}}{60\text{km/h}} \\ &= 1\frac{1}{13} \text{ h}\end{aligned}$$

(b) The approximate time at which she arrives home is 1 100 hours

We hope that after comparing your answers with the model answers provided in the above feedback you might have got all the answers correct. Congratulations! If not then I suggest you try the questions you got wrong again. You may be required to read through the section again in order to gain more understanding. You can also discuss your answers with your tutor and other students. This is important, especially, if you still did not understand some sections after revising the relevant sections, but remember that discussion of your studies with others is always beneficial because you are able to share ideas and help one another. When you are happy with your progress, you can move on to the next topic.



Answers to
Assignment 2

Answers to Topic Exercise 2

We hope you found this activity interesting and challenging. We suppose that you got more than 75% of total number of answers correct, if not then we suggest that you study this section again.

$$\begin{aligned}
 1. \quad (a) \quad \text{acceleration} &= \frac{\text{change in velocity}}{\text{time taken}}; \text{ where velocity} = 20 \text{ m/s,} \\
 & \hspace{15em} \text{Time} = 15 \text{ s} \\
 &= \frac{20 \text{ m/s}}{15 \text{ s}} \\
 &= \frac{4}{3} \\
 &= 1\frac{1}{3} \text{ m/s}^2
 \end{aligned}$$

Therefore, the acceleration is $1\frac{1}{3} \text{ m/s}^2$

$$\begin{aligned}
 (a) \quad \text{Distance} &= \text{area under graph} \\
 &= \frac{1}{2}(a + b) \times h; \text{ where } a = 25 \text{ s, } b = 40 \text{ s, } h = 20 \text{ m/s} \\
 &= \frac{1}{2}(15 + 40) \times 20 \\
 &= \frac{1}{2} \times 55 \times 20 \\
 &= 550 \text{ m}
 \end{aligned}$$

Therefore, the distance covered in 40 seconds is **650 m**

$$(b) \quad \text{Average speed} = \frac{\text{total distance}}{\text{total time taken}}$$

Where distance = area under graph

$$\begin{aligned}
 &= \frac{1}{2}(a + b) \times h; \text{ where } a = 25 \text{ s, } b = 50 \text{ s, } h = 20 \text{ m/s} \\
 &= \frac{1}{2}(15 + 40) \times 20 \\
 &= \frac{1}{2} \times 55 \times 20 \\
 &= 550 \text{ m}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{Average speed} &= \frac{550 \text{ m}}{40 \text{ s}} \\
 &= 13.75 \text{ m/s}
 \end{aligned}$$

Therefore, the average speed for the whole journey is **13.75 m/s**

$$\begin{aligned}
 2. \quad (a) \quad \text{acceleration} &= \frac{\text{change in velocity}}{\text{time taken}}; \text{ where velocity} = 15 \text{ m/s,} \\
 & \hspace{15em} \text{Time} = 5 \text{ s} \\
 &= \frac{15 \text{ m/s}}{5 \text{ s}} \\
 &= 3 \text{ m/s}^2
 \end{aligned}$$

Therefore, the acceleration is **3 m/s²**

$$\begin{aligned}
 (b) \quad \text{Distance} &= \text{area under graph} \\
 &= \frac{1}{2}(a + b) \times h; \text{ where } a = 15 \text{ s, } b = 10 \text{ s, } h = 15 \text{ m/s}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} (15 + 10) \times 15 \\
 &= \frac{1}{2} \times 25 \times 15 \\
 &= 187.5 \text{ m}
 \end{aligned}$$

Therefore, the distance covered in 15 seconds is **187.5 m**

$$(c) \text{ Acceleration} = \frac{\text{change in velocity}}{\text{time taken}};$$

Where change in velocity = 0 m/s, time = 10 s.

$$\begin{aligned}
 &= \frac{0 \text{ m/s}}{10 \text{ s}} \\
 &= 0 \text{ m/s}^2
 \end{aligned}$$

Therefore, the rider is not accelerating since the speed is constant.

$$(c) \text{ Retardation} = \frac{\text{change in velocity}}{\text{time taken}}$$

Time taken = $\frac{\text{change in velocity}}{\text{retardation}}$; change in velocity = 15 m/s,

$$\begin{aligned}
 \text{Time taken} &= \frac{15 \text{ m/s}}{\frac{1}{3} \text{ m/s}^2} \\
 &= \frac{15 \times 3}{1} \\
 &= 45 \text{ s}
 \end{aligned}$$

Therefore, the time taken after 20 seconds of retardation is **45 s**

$$\begin{aligned}
 (d) \text{ Distance} &= \text{area under graph} \\
 &= \frac{1}{2} (a + b) \times h; \text{ where } a = 15 \text{ s, } b = 20 + 45 = 65 \text{ s,} \\
 & \hspace{15em} H = 15 \text{ m/s} \\
 &= \frac{1}{2} (15 + 65) \times 15 \\
 &= \frac{1}{2} \times 80 \times 15 \\
 &= 600 \text{ m}
 \end{aligned}$$

Therefore, the distance covered before coming to rest is **600 m**

$$3. (a) \text{ Retardation} = \frac{\text{change in velocity}}{\text{time taken}};$$

where velocity changes from 50 m/s to 20 m/s, time taken = 10 s

$$\begin{aligned}
 \text{Retardation} &= \frac{50 - 20}{10} \\
 &= \frac{30}{10} \\
 &= 3 \text{ m/s}^2
 \end{aligned}$$

Therefore, the initial retardation is **3 m/s²**

$$(b) \text{ Acceleration} = \frac{\text{change in velocity}}{\text{time taken}},$$

Where velocity changes from 20 m/s to 50 m/s, time changes from 25 s to 30 s.

$$\begin{aligned} \text{Acceleration} &= \frac{50 - 20}{30 - 25} \\ &= \frac{30}{5} \\ &= \mathbf{6\text{m/s}^2} \end{aligned}$$

Therefore, the acceleration is $\mathbf{6\text{ m/s}^2}$

(c) When $t = 28$, the rider was not retarding but accelerating according to the diagram,

So retardation = - acceleration

$$\begin{aligned} &= - \frac{\text{change in velocity}}{\text{time taken}} \\ &= - \frac{30}{5} \\ &= - \mathbf{6\text{ m/s}^2} \end{aligned}$$

Therefore, the retardation when $t = 28$ is $- 6\text{ m/s}^2$

(d) Given: Total distance = 1km

We convert 1km to metres as 1000m

Therefore, area = 1 000m

Total area = area of first trapezium + area of rectangle + area of second trapezium + area of triangle

$$\frac{1}{2} \times 10 (50 + 20) + (20 \times 15) + \frac{1}{2} \times 5 (50 + 20) + \frac{1}{2} \times 50 \times x = 1000$$

$$5 (70) + 300 + \frac{1}{2} \times 5 \times 70 + 25x = 1000$$

$$350 + 300 + 175 + 25x = 1000$$

$$825 + 25x = 1000$$

$$25x = 1000 - 825$$

$$25x = 175$$

$$x = 7$$

$$\therefore \text{Total time} = 30 + 7 = \mathbf{37\text{ Seconds.}}$$

We hope that after comparing your answers with the model answers and working provided in the above feedback you might have got all the answers correct. Congratulations! If not then we suggest you try the questions you got wrong again. You may be required to read through the section again in order to gain more understanding.

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Unit 8

Symmetry

Introduction

Welcome to Unit 8 of your Mathematics Grade 11 course. Unit 8 discusses symmetry in two dimensions. In the previous unit you learnt about travel graphs. You learnt how to interpret distance-time and speed-time graphs. You also learnt how to calculate the distance covered in a specified time from a given speed-time graph. You further learnt about acceleration in relation with these graphs. There is no direct link between travel graphs and symmetry. Nevertheless both are essential for successful completion of the Grade 11 Mathematics course.

In this unit you will learn about lines of symmetry as well as rotational symmetry. For now we can say that a line of symmetry divides a figure into two identical parts and a figure has rotational symmetry if it coincides with itself at least twice in 360° .

This unit has three topics. In the first topic you will learn about lines of symmetry. You will learn how to determine the number of lines of symmetry of plane figures. In the second topic you will learn about rotational symmetry. You will also learn how to determine the order and magnitude of rotational symmetry of a plane figure. In the third topic you will learn about things in the natural and built environments that are symmetrical and those that can be said to have rotational symmetry.

Upon completion of this unit you will be able to:



Outcomes

- *Identify* lines of symmetry in a two dimensional shape.
- *Identify* rotational symmetry in two dimensional shapes.
- *Determine* the order of rotational symmetry in two dimensional shapes
- *Determine* the magnitude of rotational symmetry in two dimensional shapes
- *Identify* symmetry in built and in the natural environment.

Time Frame



How long?

The estimated time for you to complete this unit is 5 to 9 hours. This estimated time include the time you will spend reading as well as the time you will be doing activities and checking them against the feedback. Since we do not all learn at the same pace do not feel discouraged if you do not finish within the estimated time.

In addition to the above mentioned time you will also spend some time doing the end of topic exercises. We encourage you to spend 30 minutes on each of the three topic exercises. This brings the total time to spend on completing this unit to between $6\frac{1}{2}$ and $10\frac{1}{2}$ hours.

Learning Resources



Tip

You need the following resources if you are to understand this unit with minimum difficulties.

- Ruler
- Cardboard
- Pins
- Pencil or pen

Teaching and Learning Approaches



Tip

In this unit we have used teaching methods that will help you to study effectively in several ways as explained below. These methods are:

- **Conceptual:** This method will help you understand facts, rules and procedures. Included in this unit are a number of diagrams to help you understand facts, rules and procedures.
- **Problem solving:** This method will help you to solve mathematical problems relating to real life situations. It will also help you discuss mathematical problems with others. The examples and activities in the unit will help you develop problem solving skills.

- **Skills:** This method will help you practice using the facts, rules and procedures by doing self-marked activities and topic exercises.

The unit has got several activities as well as end of topic exercises. You are encouraged to do these exercises and activities because as you do so you will be putting into practice the three teaching and learning methods. You will thus achieve greater understanding of the unit and be able to relate the knowledge to real life situations.



Terminology

Anticlockwise:	Moving around in the opposite direction to the movement of the hand of a clock.
Clockwise:	Moving around in the same direction as the hands of a clock.
Complete Turn:	An angle of 360° .
Line of symmetry:	A line that can be drawn through a plane figure so that the figure on one side is the reflection image of the figure on the opposite side.
Outline:	The line that goes around the edge of something.
Plane:	A plane is a flat surface made up of points that has no depth and extends indefinitely in all directions.
Rotational Symmetry	If a figure can be rotated less than 360° about a point so that the figure the image and pre-image are indistinguishable then it has rotational symmetry.

Topic 1 Lines of Symmetry

Introduction

This is the first of the three topics in this unit. The material covered will address the first of the five unit outcomes namely: identify lines of symmetry in a two dimensional shape. You may have come across the word symmetry. The word symmetry refers to the exact match in size and shape between two halves, parts or sides of something.

In this unit the material discussed will help you to determine the number of lines of symmetry that a plane figure has. For this we refer you to what you studied in the junior secondary course. You will be able to know the properties that enable a figure to have a line or lines of symmetry.

Upon completion of this topic you will be able to:

- *Identify* line symmetry in a two dimensional shape.



Outcome

Lines of Symmetry

In your Junior Secondary School Course you learnt about lines of symmetry. You learnt that a line of symmetry divides a plane figure into two identical parts. You also learnt that a plane figure is a flat shape. Further, you learnt how to determine whether or not a figure has a line or lines of symmetry. You also learnt how to draw lines of symmetry on a given figure.

In this topic we are going to discuss the same concepts covered at junior level. So the material that will be covered in this topic will not be entirely new to you.

We will begin by revising the subject on lines of symmetry; discuss how to identify a figure with a line of symmetry and those that have more than one line of symmetry. Having said that, we require you to do the following activity by applying the knowledge you acquired from your previous studies at the junior course.



Activity 1

Does the shape in Figure 1 have a line of symmetry? If so draw a line of symmetry on the diagram.

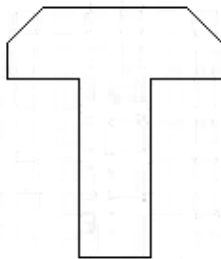


Figure 1: First revision item on line of symmetry



Feedback

I trust you were able to provide the correct answer for this activity based on your previous knowledge. You would be correct if you said that the figure has one line of symmetry. In Figure 2 the line drawn joining the two points A and B is the line of symmetry for this particular figure.

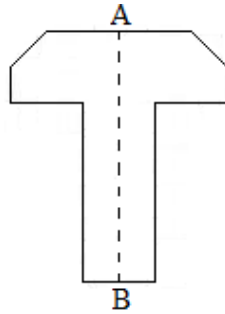


Figure 2: Explanation 1 of Figure 1

The line AB divides the figure into two identical shapes. The shapes will appear as the following Figure 3 shows:

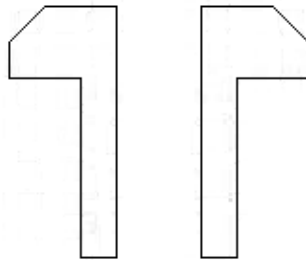


Figure 3: Explanation 2 of Figure 1

In the activity you have just found that line AB is a line of symmetry for this particular shape. To continue our discussion on lines of symmetry let us now discuss the kite ABCD below.

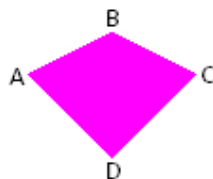


Figure 4: A Kite

Which two points can be joined by a line so that the line divides the kite into two parts such that each part is the mirror image of the other? A

mirror image of an object is like a reflection of that object. You will realise that a line joining B and D divides the kite into identical halves—triangles ABD and BCD.

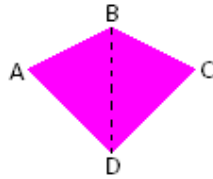


Figure 5: Division of a kite into identical halves

The result is two identical triangles ABD and BCD.

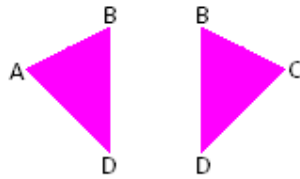


Figure 6: Division of a kite into two identical triangles

The imaginary line BD is called the **line of symmetry** or **axis** of this particular shape.

Shapes with lines of symmetry have line **symmetry** and are said to be **symmetrical**. Any symmetrical shape can be divided into two identical halves along an imaginary line.

We have so far discussed what a line of symmetry is and we also demonstrated through diagrams how this is represented. Can an object have more than one line of symmetry? This question will be answered in the next section as we continue addressing the first unit outcome.

Number of Lines of Symmetry

Some plane figures have more than one line of symmetry. Can you think of some of these? A square and a rectangle are some of the shapes that can have more than one line of symmetry. In the following activity you will be required to draw the lines of symmetry. The activity will give you a chance to apply what you have learned so far. It is not a difficult activity. So ensure that you complete it and then compare your answer with the feedback.



Activity 2

Figure 7 shows a square and a rectangle. Both shapes have more than one line of symmetry.

Copy them and draw all the lines of symmetry of the square and the rectangle.



Figure 7: Square and Rectangle

Use the space that follows.



Feedback

I told you that both shapes have more than one line of symmetry. I trust you were able to draw all the lines of symmetry for the square.

*The square is symmetrical and it has 4 lines of symmetry
The lines numbered 1 to 4 in Figure 8 are the lines of symmetry for this particular square.*

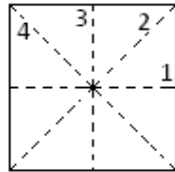


Figure 8: Illustration of lines of symmetry for the square in Figure 7

If the square is split along line 1 the resulting rectangles would appear as Figure 9 shows.



Figure 9: Division of the square into identical rectangles

If the square is split along line 2 the resulting triangles would appear as Figure 10 shows.

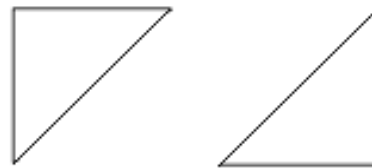


Figure 10: Division of the square into identical triangles

If the square is split along line 3 the resulting rectangles would appear as Figure 11 shows.



Figure 11: Division of the square into identical rectangles

If the square is split along the line 4 the resulting triangles would appear as Figure 12 shows.

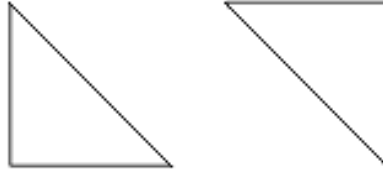


Figure 12: Division of the square into identical Triangles

The rectangle is also symmetrical. It has 2 lines of symmetry

The lines numbered 1 and 2 in Figure 13 are the lines of symmetry for this particular rectangle.

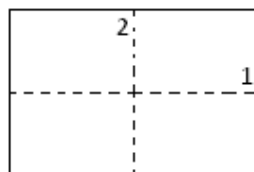


Figure 13: Illustration of lines of symmetry for the rectangle in Figure 7

If the rectangle is split along line 1 the resulting rectangles would appear as Figure 14 shows.

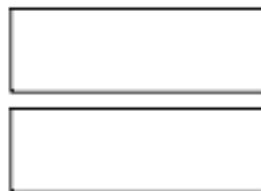


Figure 14: Division of the rectangle into identical rectangles

If the rectangle is split along line 2 the resulting rectangles would appear as follows:



Figure 15: Division of the rectangle into identical rectangles

This activity has shown that rectangles and squares have more than one line of symmetry. Other shapes also have more than one line of symmetry. Consider the following example.

Example 1

Let us find out how many lines of symmetry the figure below has.

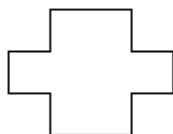


Figure 16(a): Third discussion item with more than one line of symmetry

Solution

The figure has 2 lines of symmetry

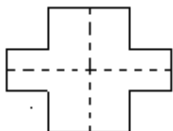


Figure 16(b): Third discussion item with more than one line of symmetry

We have so far discussed that lines of symmetry divide a plane figure into two identical figures. Let us now do a practical activity. The activity will consolidate your knowledge of what a line of symmetry does to a plane figure,



Practical Activity

You will need paper, a pair of scissors, a pencil and a ruler.

Cut out from a paper a square with sides of length 15cm. Fold the square in such a way that the two parts overlap each other. Do not fold it more than once at any given time. Do this in as many ways as it is possible.

You then are required to answer the following:

- State how many ways this is possible?
- Is it along the folds that we find the lines of symmetry?
- How many such exist for the square?



Feedback

You must have folded the square in four ways. If you did not try again until you have done so in four ways. The broken lines in Figure 18 indicate where you should fold.

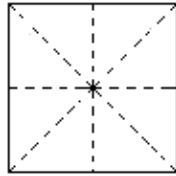


Figure 17: Illustration of the answer to the Practical Activity

You must have realised that where the square was folded is where the lines of symmetry pass.

In the activity it is along the folding that we find the lines of symmetry. This activity has shown that you can fold a plane figure along a line of symmetry so that the two sides overlap each other.

Our discussion of the first topic ends here. Now let us briefly review what we have discussed.

Topic 1 Summary



Summary

Topic 1 of unit 8 discusses lines of symmetry. In this topic you learnt that certain figures have imaginary lines called lines of symmetry. Each line of symmetry divides an object into two identical parts. You learnt how to identify lines of symmetry and did activities on drawing them. You learnt that objects which have lines of symmetry are said to be symmetrical. To help us achieve our objective we showed you some examples of plane figures that have lines symmetry. You also did activities to help you understand the topic.

You are encouraged to do the end of topic exercise which can be found at the end of the unit. Thereafter compare your solutions with the feedback which follows the exercise. It is very important that you first answer the questions in the exercises and then compare your solutions to those in the feedback. If you found some parts of the exercise difficult revise the relevant sections of the topic before moving on to Topic 2.

We will discuss rotational symmetry in the second topic.

Topic 2 Rotational Symmetry

Introduction

This is the second topic in this unit on symmetry. Unlike the first topic the material discussed in this topic is relatively new to you. You are going to learn what rotational symmetry is. Knowledge of rotational symmetry will make it more interesting to look at things with rotational symmetry. You will also be able to understand other topics on geometry. For example in your Grade 12 Mathematics course, you will study a unit on transformations. In this unit there is a topic on rotation. The knowledge you will acquire now will be of great benefit to you as you study the topic on rotation.

This topic will focus on the second, third and fourth outcomes of the unit.

Upon completion of this topic you will be able to:



Outcomes

- *Identify* rotational symmetry in a two dimensional shape.
- *Determine* the order of rotational symmetry in two dimensional shapes
- *Determine* the magnitude of rotational symmetry in two dimensional shapes

What is rotational symmetry?

To adequately understand what rotational symmetry is you have to know the meaning of two words namely 'rotate' and 'rotation'. The word 'rotate' means to move or turn around a fixed point while the word 'rotation' refers to the action of moving in a circle around a central fixed point.

If a figure can be rotated less than 360° about a point so that it fits its outline, then the figure has rotational symmetry. The point about which it rotates is called the **centre of rotational symmetry**.

In this section we will consider what happens when a figure is rotated about point 360° .

We begin our discussion by considering Figure 1 below.

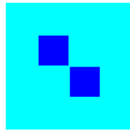


Figure 1: First discussion item on rotational symmetry

Suppose it is rotated about its centre in an anticlockwise direction (opposite to the movement of the hand of a clock). When you examine the diagram below you will notice that after the figure is rotated 180° about its centre it appears as though it has not moved; it coincides with itself after 180° . This shows that the figure has rotational symmetry.

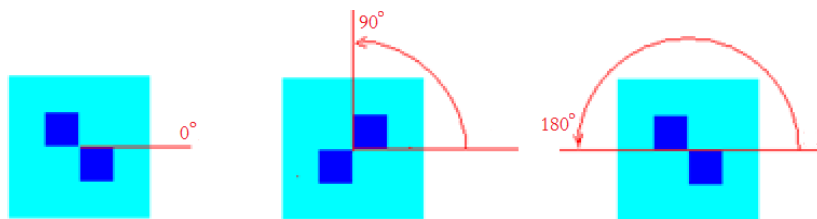


Figure 2: Rotating Figure 1 in an anticlockwise direction

Consider also the figure below. Suppose it is rotated in a clockwise direction (moving in the same direction as that of the hands of a clock).

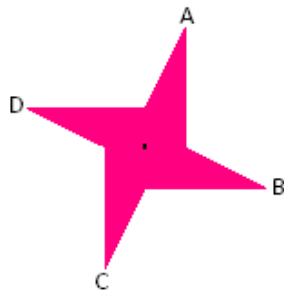


Figure 3: Second discussion item on rotational symmetry

Notice the positions of A, B, C and D when it is rotated in the clockwise direction as shown in the diagrams that follow. The figure coincides with itself four times in 360° .

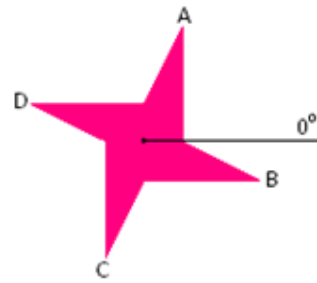


Figure 4: Figure 3 before rotation takes place

After rotating an angle of 90° the figure coincides with itself and appears as though it has not moved. Figure 6 illustrates this.

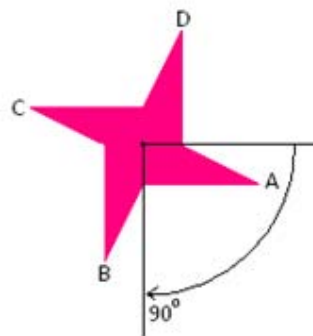


Figure 5: A 90° clockwise rotation of Figure 3

The figure coincides with itself again after rotating 180° from its original position. You can see its position as illustrated by figure 7.

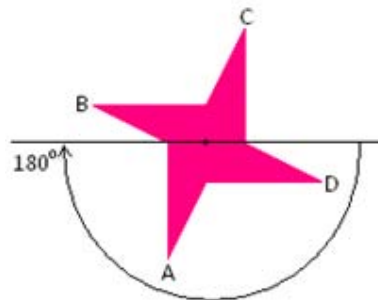


Figure 6: A 180° clockwise rotation of Figure 3

Figure 8 shows the shape after it has rotated an angle of 270° .

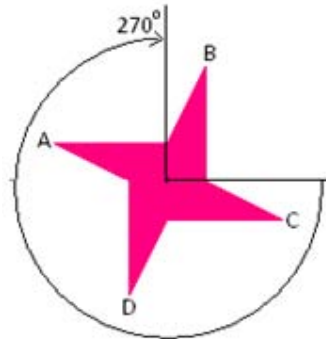


Figure 7: A 270° clockwise rotation of Figure 3

A 360° anticlockwise rotation takes the figure to its original position.

The number of times and angle in which the shapes fit their outlines are not the same for different shapes. This brings us to the order and magnitude of rotational symmetry. We will first look at the order of rotational symmetry.

Order of Rotational Symmetry

The order of rotational symmetry of a plane figure is the number of times the figure fits its outline in 360° when it is rotated about its centre of rotation. In discussing the order of rotational symmetry let us find out how many times the figure below (Figure 8) fit its outline when it is rotated 360° in the clockwise direction about its centre?

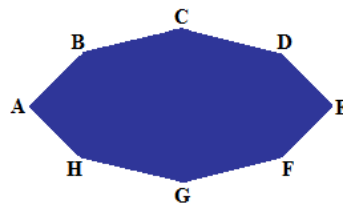
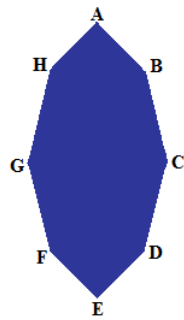


Figure 8: Discussion item on order of rotational Symmetry

The figure will be in the position as shown below when it is rotated 90° clockwise about its centre.



In this position the figure's outline does not coincide with its original position.

Figure 9: A 90° clockwise rotation of Figure 8

When it rotates 180° it fits its outline. Notice the positions of the different vertices (point where two sides meet) in Figures 10 and 11 below.

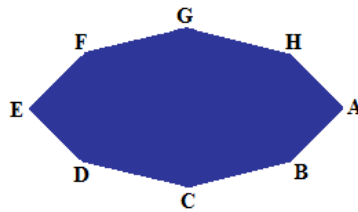


Figure 10: A 180° clockwise rotation of Figure 8

When it rotates 360° it fits its outline.

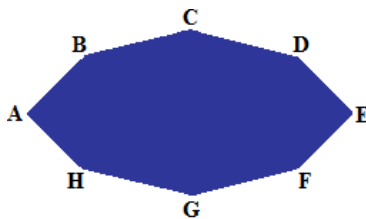


Figure 11: A 360° clockwise rotation of Figure 8

You can see from Figures 8, 9, 10 and 11 above that the figure coincides with itself after 180° and this happens twice in 360° . Therefore, the order of rotational symmetry is 2 since it fits its outline twice in 360° .

Let us now turn to magnitude of rotational symmetry.

Magnitude of Rotational Symmetry

Consider, once more the figure below which we had in our earlier discussion on rotational symmetry.

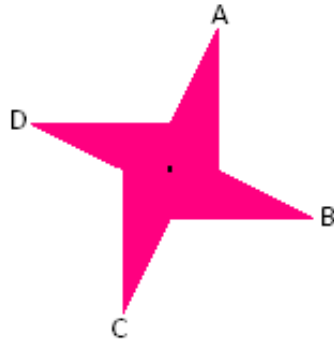


Figure 12: First discussion item on magnitude of rotational symmetry

After rotating an angle of 90° the figure fits its outline.

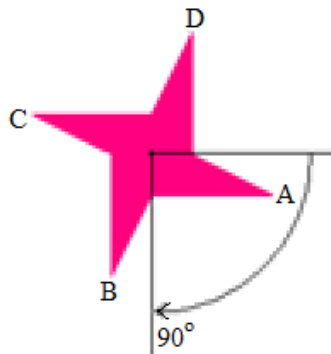


Figure 13: A 90° clockwise rotation

After rotating another 90° the figure fits its outline, figure 14 below.

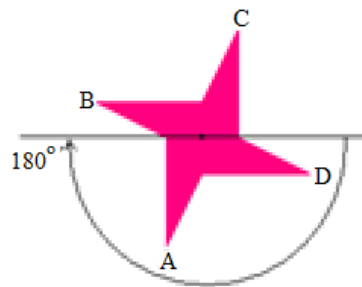


Figure 14: A 180° clockwise rotation

After rotating another 90° as shown in Figure 15 the figure fits its outline

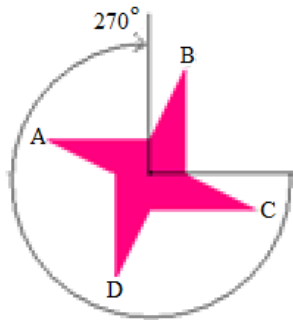


Figure 15: A 270° clockwise rotation

This figure fits its outline every 90° . Therefore it has a magnitude of rotational symmetry of 90° .

A figure has rotational symmetry if it can be rotated 180° or less clockwise or counter clockwise about a point so that the image and the pre - image coincide.

A square has a rotational symmetry of magnitude 90° and order of 4. Let us do the following practical activity to confirm this. You are encouraged to do it before you look at the feedback as it will strengthen your understanding.



Practical Activity

To do this activity you need a cardboard, a pin, a pencil and a ruler.

Cut out a square from a piece of cardboard with each side 10cm long. Mark the corners of the square with numbers 1, 2, 3 and 4. Draw diagonals on the square. Get another cardboard square with sides of at least 15cm long. Fasten the two squares with a pin at the point where the two diagonals meet. Mark out the position of the square by tracing the outline of the smaller board on the bigger one. Write the corresponding numbers on the bigger board also. The boards should look as illustrated in Figure 16.

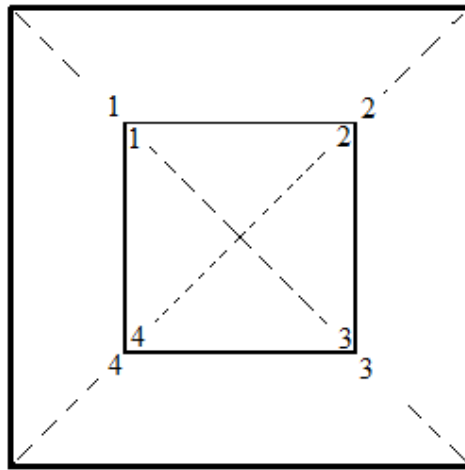


Figure 16(a): Two square cardboards of different sizes

Now that you have made the boards and fastened them together rotate the smaller square in the clockwise direction and count how many times the square fits its marked outline in 360° .



Feedback

As you were rotating the smaller square cardboard you must have counted the number of times it coincides with its outline. You must have noticed that this happens four times. From its original position the square fits its outline every 90° .

When you rotate the square it coincides with itself after 90° . Look at figure 16(b). Observe that a 90° clockwise rotation moves 1 to the original position of 2.

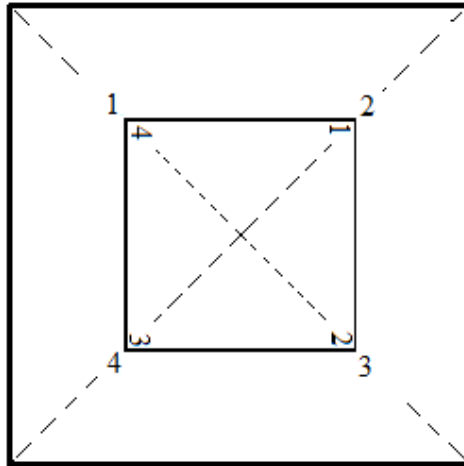


Figure 16(b): Two square cardboards of different sizes at 90°

When you rotate the square an angle of 180° 1 moves to the original position of 3 and the square fits its outline. Look at figure 16(c).

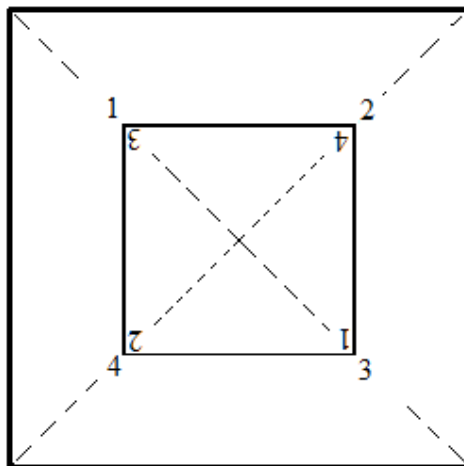


Figure 16(c): Two square cardboards of different sizes at 180°

A further 90° clockwise rotation moves 1 to the original position of 4 and the square fits its outline as figure 16(d) illustrates.

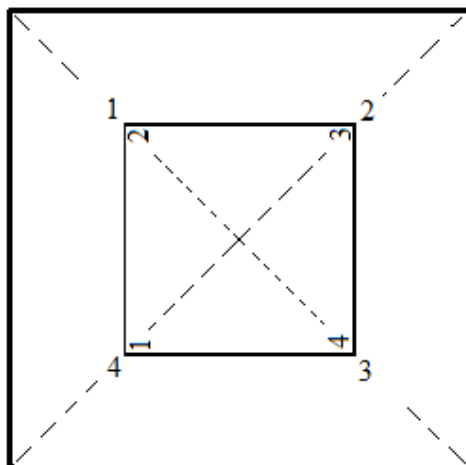


Figure 16(d): Two square cardboards of different sizes at another 90°

A complete turn takes 1 to its original position.

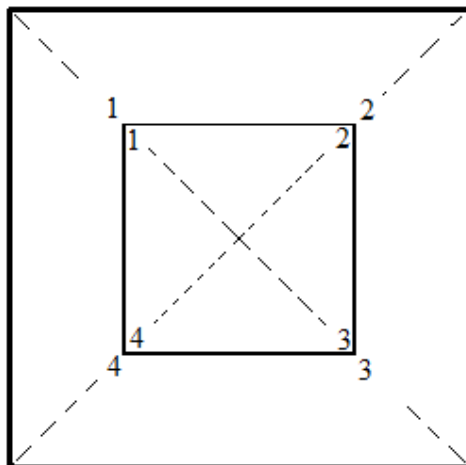


Figure 16(e): Two square cardboards of different sizes –complete turn

If you check the movement of all the other numbers you will notice that they move in a similar manner as 1. This practically shows that a square has rotational symmetry of order 4 and magnitude of 90° .

We have so far discussed what rotational symmetry is. We have also discussed the order of rotational symmetry as well as magnitude of rotational symmetry. We now give you a summary of what we have covered in this topic.

Topic 2 summary



Summary

In this topic you have learnt about rotational symmetry. We have defined the words and you learnt that rotational symmetry has order as well as magnitude. We gave examples of plane figures that have rotational symmetry. We also discussed the order and magnitude that different plane figures have. The definitions of order of rotational symmetry and magnitude of rotational symmetry were given. To achieve our objective we gave you examples of plane figures that have rotational symmetry. You have also done a few activities to help you practice.

Topic 2 exercise in the assignment section is also part of our strategy to achieve our objective. Complete this exercise and correct your answers as you have done with the previous topic.

The next topic which is the third is very much related to the first two. The third topic will discuss things in the natural and built environments which exhibit symmetry.

Topic 3 Symmetry in Built and in the Natural Environment

Introduction

As the title of the topic suggests this topic discusses symmetry in relation to the environment. We are going to look at ways in which symmetry is exhibited in the built and natural environments. We will give examples of how designs of buildings, patterns and animal life are made to appear symmetrical. We will also show how certain shapes in our built environment and in the natural environment can be said to have rotational symmetry.

Upon completion of this topic you will be able to:



Outcome

- *Identify* symmetry in built and in the natural environment.

Symmetrical Objects in Our Built and Natural Environment

In Topic 1 we defined symmetry and discussed lines of symmetry. You learnt that a line of symmetry divides a plane object into two halves each identical to the other. You also learnt that each side of the line of symmetry looks like a reflection of the other. It was also mentioned to you that plane figures with lines of symmetry are said to be symmetrical. Many things in our environment are symmetrical. These include buildings, flowers and animal life. Of course a single line cannot divide a three dimensional object. With three dimensional objects we talk of **plane symmetry**.

Consider the following cuboid in Figure 1. A cuboid is a solid object which has six rectangular sides at right 90° to each other.

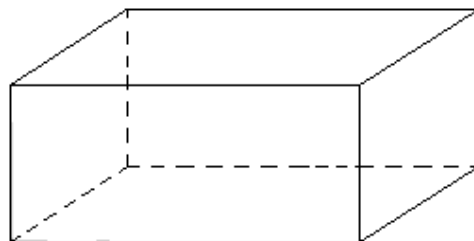


Figure 1(a): Cuboid for first discussion item on symmetrical objects

There exists several planes at which the cuboid can be divided into two such that one side is like a reflection of the other. One of them is the one illustrated in the following diagram.

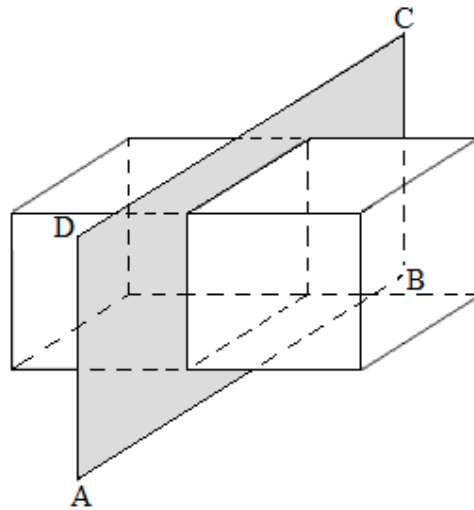


Figure 1(b): Cuboid division – plane of division

The plane ABCD divides the cuboid into two identical parts each of which is like the mirror image of the other. Therefore the cuboid has plane symmetry and it is symmetrical.

Many designers have come up with different structures that have symmetry. Look at the drawing of a building in Figure 2 below.

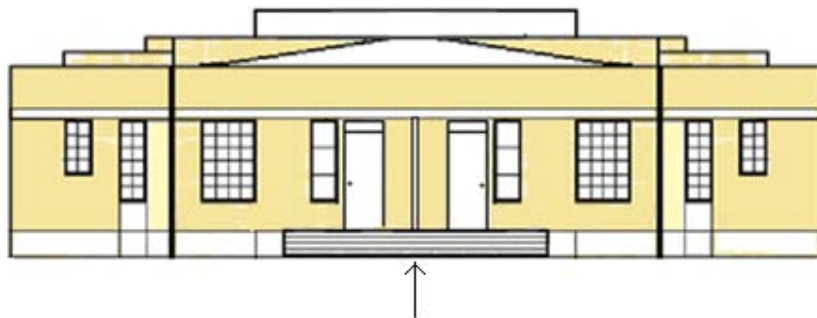


Figure 2: a symmetrical building

One side of the building looks like the reflection of the other. It is in that sense that we say it is symmetrical. One side of the arrow looks like the reflection of the other.

Here is another picture that shows part of a building which is symmetrical.



Figure 3: Another symmetrical building

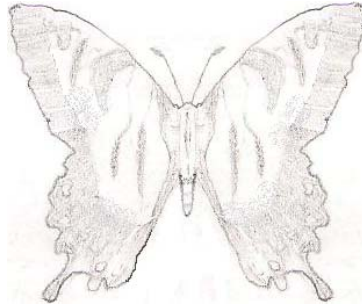
Source: fineartamerica.com

The building in the picture can be said to be symmetrical because one side of the building is like a reflection of the other side. Many buildings are designed to look that way.

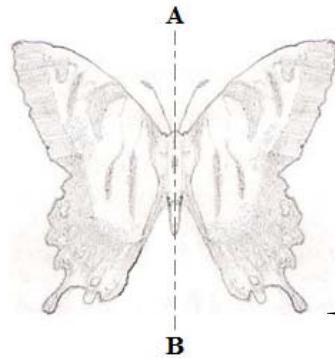
What about the natural environment? Can we find symmetry there? For sure we can.

Symmetry is not just restricted to the built environment but we can find it even in the natural environment. Can you think of living things that appear symmetrical?

Look at the picture of the butterfly below.

Example**Figure 4(a): Sketch of a butterfly**

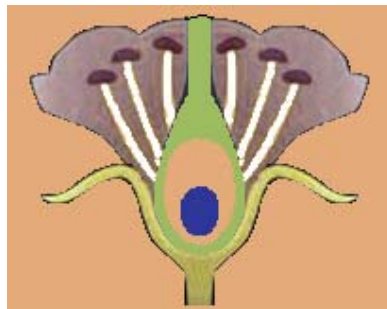
This butterfly can be divided into two identical parts. This is illustrated in the following picture.



The butterfly is symmetrical across AB

Figure 4(b): Sketch of a butterfly divided into two identical parts

Plants too are symmetrical as the drawing of the cross section of a flower below shows.

**Figure 5(a): cross section of a flower**

You can now see where the plane of symmetry passes.

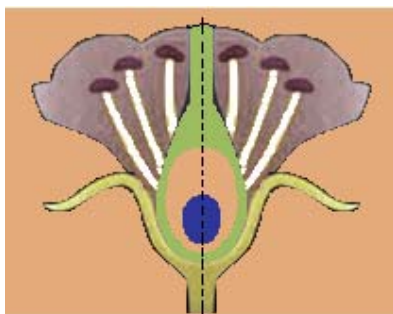


Figure 5(b): cross section of a flower with plane of symmetry

Apart from plants even in animals we find symmetry. Here is an example to show this. Look at the picture of a flying insect.

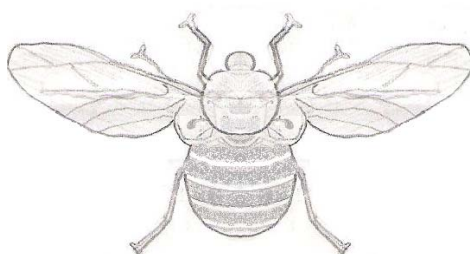


Figure 5(c): flying insect

You must realise that there are many more plants, insects and animals which are symmetrical that you have seen where you live or in books, videos and on television. So you can see that knowledge of symmetry makes viewing nature even more interesting.

Do you recall that at the beginning of this unit we informed you that you are going to learn about things in the natural and built environments that are symmetrical and those that can be said to have rotational symmetry? Let us now turn our attention to discussing rotational symmetry in our built and natural environment.

Rotational Symmetry in Manmade and in the Natural Environment

Rotational symmetry is all around us because almost everywhere you go you will find it. Let us consider patterns in carpets or mats.

Some patterns used to decorate mats and carpets have rotational symmetry. Such patterns make carpets and mats interesting to look at. The pattern in the picture below is an example of such patterns.



Figure 6: a pattern with rotational symmetry

Below is a picture of floor tiles with patterns having rotational symmetry.

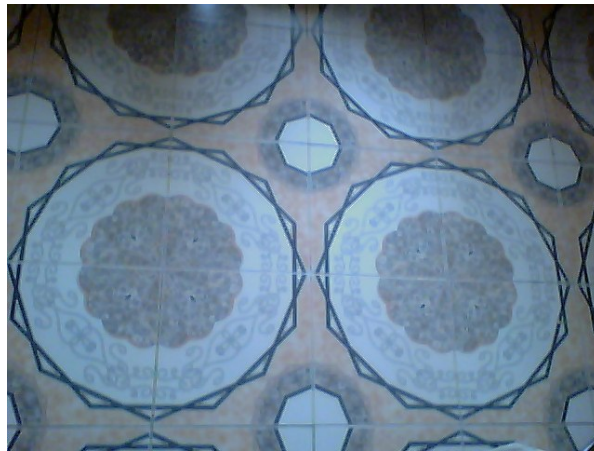


Figure 7: floor with patterns with rotational symmetry

Rotational symmetry in our built environment is not limited to patterns on floors. Some structures are designed with rotational symmetry as the following picture illustrates.

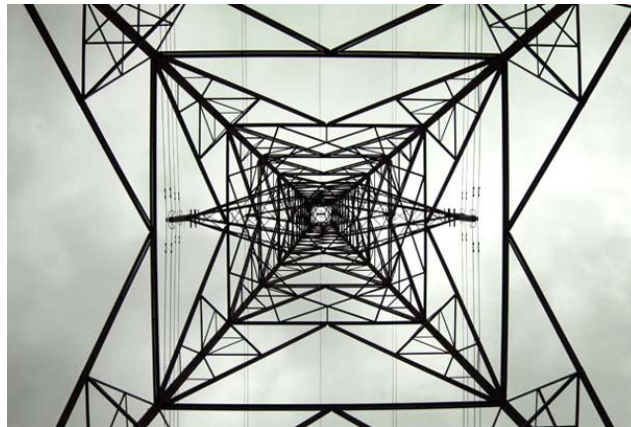


Figure 8: A pylon with rotational symmetry

The following picture shows a flower with rotational symmetry. When the flower is rotated about the centre it appears as though it has not moved.



Figure 9: A flower with rotational symmetry

In homes we find mats and carpets with patterns which have rotational symmetry. The following shape can be used to decorate a mat or carpet. When the shape is rotated about its centre it coincides with its original position after a certain angle and appears as though it has not moved. Look at it carefully. You will notice that it has rotational magnitude of 90° because it coincides with itself after 90° .

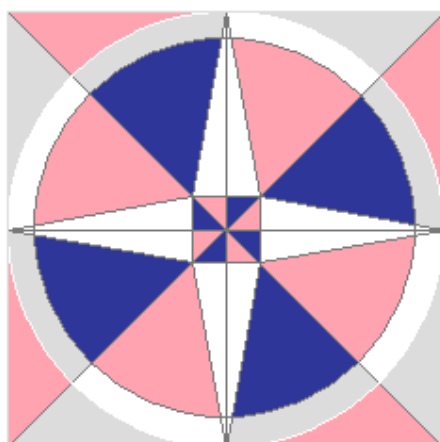


Figure 10: pattern with rotational symmetry

There are many more things in our surroundings that exhibit some kind of symmetry. Now that you have learnt about symmetry you are encouraged to be observant because by doing so you will appreciate the design of many things in your environment. The following is a summary of Topic 3.

Topic 3 Summary



Summary

In Topic 3 you have learnt that in our built and natural environments there is symmetry. You learnt that in the built environment and in the natural environment there are things that are symmetrical and those that have rotational symmetry. You were given examples of these, mostly through drawings. Some of the drawings you viewed were those of buildings, plants and insects.

You are encouraged to do Topic Exercise 3 in the assignment section. Remember to use the feedback to correct your work and decide whether to revise some parts of the topic or not.

The unit summary below gives you an overview of what was discussed in the unit.

Unit 8 Summary



Summary

In Unit 8 you have learnt about symmetry in two dimensions. In Topic 1 we defined symmetry and you also learnt about lines of symmetry while in Topic 2 you learnt about rotational symmetry. The last topic discussed symmetry in the built up and natural environment.

In the first topic you learnt that a line of symmetry or axis divides a plane figure in two identical parts. You also learnt that some plane figures have more than one line of symmetry. We also showed you examples of things that are symmetrical in our environments.

The second topic covered rotational symmetry. You learnt about the order and magnitude of rotational symmetry. We defined the order of rotational symmetry as the number of times an object coincides with its outline when it is rotated about its centre 360° in clockwise or anti-clockwise direction while the magnitude is the angle it takes for an object to coincide with itself. This angle is 180° or less.

In the third and last topic you saw examples of things that are symmetrical as well as of those that exhibit rotational symmetry in our built and natural environment.

This unit has no assessment. You should however ensure that you have done all the topic exercises and that your work was satisfactory before you move on to the next unit. Unit 9 is about congruence and similarity.

References

Mukuyamba A. Et al.1995. *Mathematics Grade 11 Pupil's Book, Zambian Secondary School Syllabus*, Chingola: Yorvik Publishing Limited.

Larson R. E. et al. 1998. *Geometry- An Integrated Approach*, Evanston:
McDougal Littell Inc.

Assignment



Assignment

This part has three topic exercises. Each exercise is based on a topic discussed in unit 8. You are encouraged to do the exercises before you compare your answers with the feedback provided. This will enable you to correctly assess your understanding of each topic. There is no end of unit self assessment.

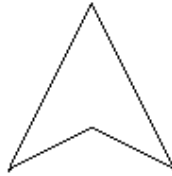
Topic 1 Exercise



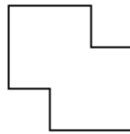
Topic 1 Exercise

1. How many lines of symmetry do the following shapes have?

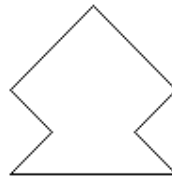
(a)



(b)

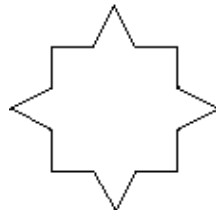


(c)

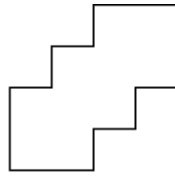


2. Draw the lines of symmetry of the following shapes:

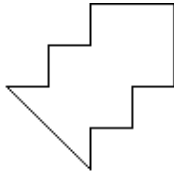
(a)



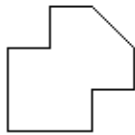
(b)



(c)



(d)



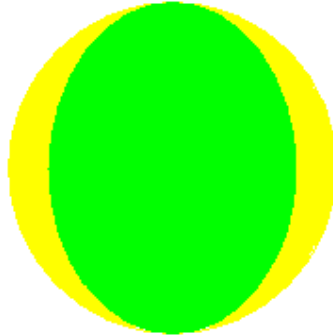
Topic 2 Exercise



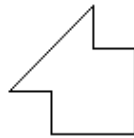
Topic 2 Exercise

State the order and magnitude of rotational symmetry for each of the following.

1.



2.



3.



4.



5.



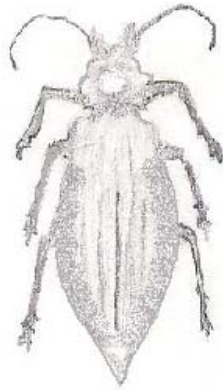
Topic 3 Exercise



Topic 3 Exercise

Comment on the symmetry of each of the following:

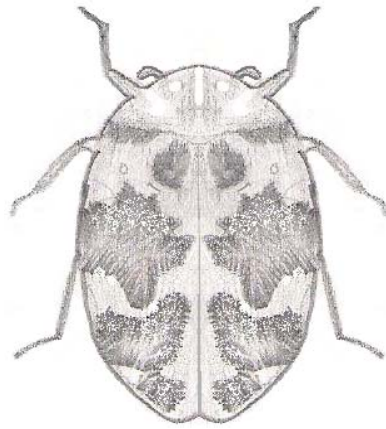
1.



2.



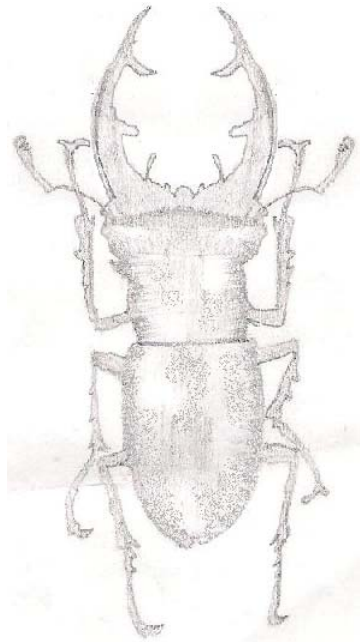
3.



4



5



Feedback

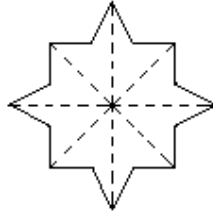
Topic 1 Exercise Feedback



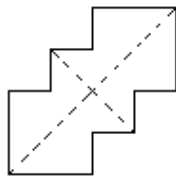
Feedback

- 1. (a) 1
- (b) 2
- (c) 1

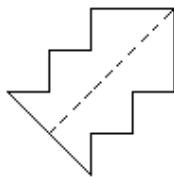
2. (a)



(b)



(c)



(d)



Topic 2 Exercise Feedback



Feedback

1. The figure has rotational symmetry of order 2 and magnitude 180° .
2. The figure has no rotational symmetry.
3. The figure has rotational symmetry of order 2 and magnitude 180° .
4. The figure has rotational symmetry of order 4 and magnitude 90° .
5. The figure has rotational symmetry of order 2 and magnitude 180° .

Topic 3 Exercise Feedback



Feedback

1. The insect is symmetrical. One side is like a reflection of the other.
2. The main structure of the building is symmetrical. One side is like a reflection of the other.
3. The insect is symmetrical. One side is like a reflection of the other.
4. The flower has rotational symmetry; if it is rotated it would appear as though it has not moved.
5. The insect is symmetrical. One side is like a reflection of the other.

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Unit 9

Congruency and Similarity

Introduction

Welcome to unit 9 in your mathematics 11 course. This means that, including this one, you now have only seven units to complete the 15 units you are going to study in grade 11. These units are very important in your Mathematics course because they are part of the questions you will be examined on at the end of your course together with other units you have learned in grade 10 and those you will learn in grade 12 in order for you to qualify for the General Certificate of Education.

In the previous Unit 8 you learnt about symmetry in two dimensional shapes. You also learnt how to determine the order of rotational symmetry in two dimensional shapes and finally you learnt how to identify symmetry inbuilt in the natural environment.

In this unit you will learn about plane figures as well as shapes in three dimensions. The knowledge of figures and solid shapes you learnt in unit 8 will be of great help in understanding figures. Unit 8 is related to unit 9 because both units use plane and solid figures.

Remember in your junior secondary school programme, you also learnt about congruency and similarity. You learnt how to identify congruency and similarity in plane figures. Here you are going to apply the same knowledge you learnt at junior secondary level although at this level we are going to include figures in three dimensions. These are figures such as cubes, Cuboids, cylinders and circles.

This unit has two topics. In each topic, you will be required to do some activities and topic exercises. You are encouraged to answer all the questions in the topic exercises.

At the end of this unit, you will be required to do the Tutor Marked Assessment 3 which you will send to your tutor for marking. In order to achieve the desired goals and the following outcomes must be met.

After studying this unit, the following outcomes should be achieved.

Upon completion of this unit you should be able to:



Outcomes

- *Identify* congruency in figures.
- *Use* congruency in solving problems requiring simple logical deductions.
- *Identify* congruency in the natural environment.
- *Find* the ratio of sides of similar figures.
- *Calculate* the unknown or angles in similar figures.
- *Identify* similarity in the natural environment.
- *Calculate* the area and volumes of similar figures.

Timeframe



How long?

We estimate that to complete this unit you will need between 11 to 15 hours. This time includes the time you will spend in doing the activities and checking them against the feedback. If you do not finish studying the unit within this estimated time do not worry since we do not all learn at the same pace.

You are encouraged to spend about 2 hours answering each topic exercise in this unit. Since there are two topic exercises in this unit this means that you will spend about four hours on these exercises. You will also have to spend two hours on the Tutor Marked Assignment which you will find after the topic exercises.

The total hours for completing the unit will thus be between 17 and 21 hours.

Learning resources



In order to study this unit with less difficulty you will need the following materials:

- Plain papers
- Mathematical instrument set.

Teaching and learning approaches



Tip

In this unit we have used four teaching and learning methods in presenting the content as in other units, for example Units 5 and 9. These methods are:

- **Interactive method:** This method requires you to work out solutions to all activities, exercises and will require you to give your answers the way you think it should be. This is how this method will be used in this unit. This will help you assess yourself whether you will be able to relate the context to real life situation.
- **Conceptual:** This method will require you understand and extract the main ideas, facts, rules, formulas and procedures used in each question and discussion in this unit. This is how this method will need you to apply to answer questions in this unit.
- **Problem-solving:** This method will require you to extract relevant information in a mathematical sentence to come up with a mathematical expression or equation. This will help you solve mathematical problems that relate to real life situation. This method will help you develop a critical analysis of real life issues. This is what you are expected to apply in solving mathematical problems.
- **Skills:** This method requires you to put into practice the use of facts, rules, formulae and procedures used in examples. This provides you with some blank spaces for you to practice to solve questions on your own by doing self-marked exercises and topic exercise. This method will improve your ability to handle any mathematical problem with less difficulty

You are required to do all the activities and/or exercises and discuss your ideas with other learners and your tutor. When you apply these methods, then it means that you will achieve greater understanding of the unit and be able to relate the knowledge to your real life situation.



Terminology

Similarity

Having the same shape but not the same size.

congruency:

Being identical in shape and size.

Topic 1 Congruence

This topic is the first of the 2 Topics in this Unit 9. In this topic you will learn how to identify congruence in figures and how to use congruency in solving problems requiring simple logical deductions. Finally, you will learn how to identify congruency in the natural environment. The topic is divided into three sections to cover the stated outcomes.

After studying the unit you will be required to answer a tutor marked assessment questions. These questions will cover questions from three units that is: 7, 8 and 9. You are encouraged not to go through the feedback before answering the topic exercises.

In this first topic of the unit, we will address the first three of the seven unit outcomes.

Upon completion of this topic you will be able to:



Outcomes

- *Identify congruency in figures.*
- *Use congruency in solving problems requiring simple logical deductions.*
- *Identify congruency in the natural environment.*

We will now proceed with the discussion of the first section where we explore ways of identifying congruency in figures. You will realise that this discussion will assist us to address the first unit outcome.

Identifying Congruency in Figures

In your area, you may have come across objects or shapes which look exactly the same, for example there are even people who look exactly alike. These are usually twins. These twins, triplets and shapes are called **identical** because they look exactly the same. .

Before going into the details of the discourse, please do the following activity to test your prior understanding of what we are discussing. This will also help in our investigation of the concept of congruence.

Activity 1



Activity

Can you list some objects or items in your environment that look exactly the same? Give your answer in the space provided below. After you have answered the question then you can compare your answers with those provided in the feed.

You can now compare your answers with the ones given below in the feedback.



Feedback

In your answer you could have listed things like certain products such as soaps, buildings such as the twin towers in USA, curtains, boxes, some types of birds just to mention but a few.

Such shapes or objects which are similar in either looks or structure are called congruent shapes or objects. The word congruent means being the same or identical in size and shape as defined in the terminology section.

You already know that the main thrust of this unit is to discuss congruency and similarity. For the purpose of our study, it is important that we should be able to know the difference between these two concepts. Once this question has been adequately addressed then the understanding of this unit will be straightforward.

The difference between the two ideas is that congruency refers to two or more figures which have exactly the same size and shape. On the other hand, the word similarity is used to describe a situation where two or more figures or shapes have corresponding sides and angles in the same ratio. Similar figures may appear the same but could be different in size.

Now let us concentrate on the discussion of how to identify congruency in figures. Congruency can be identified by comparing the sides and angles of two or more figures. If all the corresponding sides and angles are equal, then we can conclude that the figures are congruent.

I am sure you always use a plane mirror to check your appearance. If you have paid particular attention to your image which appears in a plane mirror you should have noticed that the size of your head or nose or any part of your body is exactly the same size as that which appears in a mirror. Therefore, we can simply say that, you are an object and what you see is a reflection of yourself. The reflection is what is known as an image. The image you see is congruent to an object.

Now, do the following short activity to strengthen your understanding of congruent.

Activity 2**Activity**

Why is it important to identify congruency? Write in the space provided below.

.

*Feedback*

To answer the above question you could have considered the following as examples: fittings such as the parts of a car come from many sources. For all of the pieces to fit together, they must be exactly the right size and shape.

It is important to identify congruence so that we can have the exact sizes or shapes that we need. For example the window must match the window frame exactly.

For more understanding let us consider the following polygons and see how congruency could be identified in this case.

Figure 1a and figure 1b show polygons called quadrilaterals. Quadrilaterals are four sided polygons. Please remember that polygons are any plane figures which are bounded by straight lines.

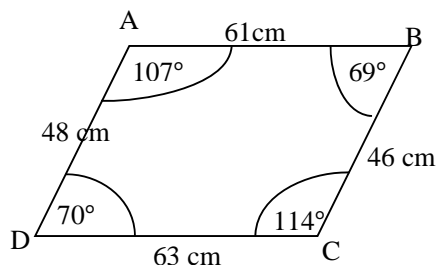


Figure 1a Quadrilateral ABCD

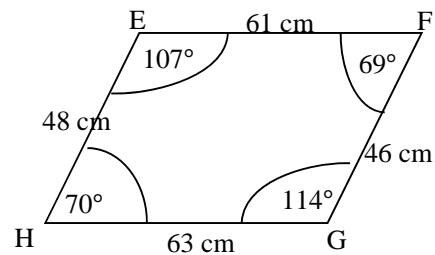


Figure 1b Quadrilateral EFGH

Let us now examine figures 1a and figure 1b to apply the knowledge of congruent. Please do the following activity to test your understanding of what we have been discussing so far.

Activity 3**Activity**

Consider figure 1(a) quadrilaterals ABCD and figure 1(b) EFGH? Use the two figures to tell which pairs of corresponding parts are congruent. Remember that the word congruent is used to describe things or figures with exact size or shape. Write your answers in the space provided below.

*Feedback*

Using the figures 1(a) and figure 1(b) we can see that the following pairs of angles are corresponding and congruent. This can be shown using symbols as shown:

$$\angle A \equiv \angle E, \angle B \equiv \angle F, \angle C \equiv \angle G, \text{ and} \\ \angle D \equiv \angle H.$$

This means that angle A is exactly the same as angle E, angle B is exactly the same as angle F, angle C is exactly the same as angle G and angle D is exactly the same as angle H.

In the expression “ $\angle A$ ” means or refers to angle A. The symbol ‘ \angle ’ stand for angle and ‘ \equiv ’ stands for exactly the same or means congruent.

We can also compare corresponding sides of the polygon as follows for sides; $\overline{AB} \equiv \overline{EF}$, $\overline{CD} \equiv \overline{GH}$, $\overline{AD} \equiv \overline{EH}$ and $\overline{BC} \equiv \overline{FG}$. The over bars for example on \overline{AB} and \overline{EF} means that the length of AB is of the same length has EF. Similarly, the length of CD is of the same degree of measurement has GH and rest goes on in the same line.

When identifying congruency in polygons, you must list the vertices of one polygon in the same order as the corresponding vertices of the other polygon.

Let us consider figure 1(a) and figure 1(b) again, polygon ABCD and polygon EFGH are congruent because all the four angles and four sides are congruent or are exactly the same.

The vertices of quadrilateral in figure 1(a) are ABCD. Vertices are points of intersection of the sides of a polygon or the faces of a solid figure.

For better understanding of what vertices are check the vertices in figure 1(a) and figure 1(b).

You will notice that vertex A in Figure 1(a) is corresponding to vertex E in Figure 1(b), vertex B is corresponding to vertex F, vertex C is corresponding to vertex G, vertex D is corresponding to vertex H. Examine the next figure to better understand what we mean by vertex.

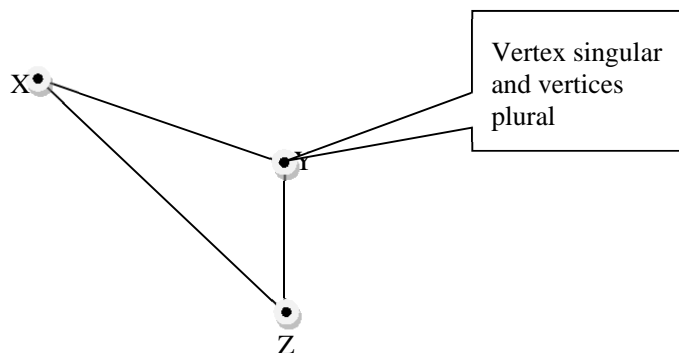


Figure 2: Vertices

Figure 2 has three vertices and these are X, Y and Z.

Now let us explore another situation to consolidate your understanding of how congruence can be identified. Figures are congruent if they match exactly when one figure is placed on top of the other.

Relate the above discussion to Figure 3 below showing triangles ABC and ABD. You will notice that A corresponds to itself, B also corresponds to itself a situation where one point corresponds to itself is said to have a **reflective property** and C corresponds to D.

This condition can be tested by superimposing one figure onto another to see whether it exactly fits its outline.

Figure 3 below shows two triangles ABC and ABD which satisfies the condition.

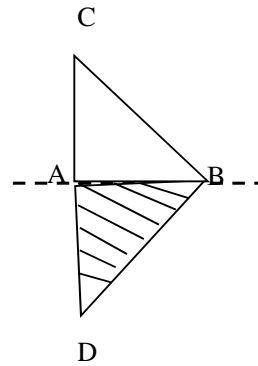


Figure 3 showing Triangles ABC and ABD

You can clearly see that the two triangles ABC and ABD can fit each other's outlines when a fold is made along AB.

Now, answer this simple question to see whether you still remember what you learnt in your junior secondary programme and what you have learnt in Unit 8 about finding the lines of symmetry in a given figure.

Activity 4



Activity

What do we call line AB? Use your previous knowledge you learnt in Unit 8 to answer this question. Write your answer in the space provided below.

*Feedback*

I am sure you still remembered what this line is called. Good! Line AB is known as the line of symmetry.

You may have noticed that there are other shapes that we have mentioned earlier which we have not discussed, but this should not worry you. Other figures and solid shapes are catered for in the properties given below.

There are two most important properties to note. These properties are true for all shapes if;



Note it

(i) Their corresponding sides are equal.

(ii) Their corresponding angles are equal. The two shapes are congruent if they have the same shape and area.

Now, do the activity below to give you more practice and deeper understanding of what we have just discussed.



Activity

Activity 5

Now, consider the two polygons given to answer the following questions.

1. List all the corresponding parts of quadrilateral JKLM and PRQS if $JKLM \equiv QSPR$.

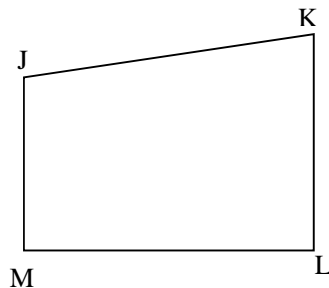


Figure 4 (a) quadrilateral JKLM

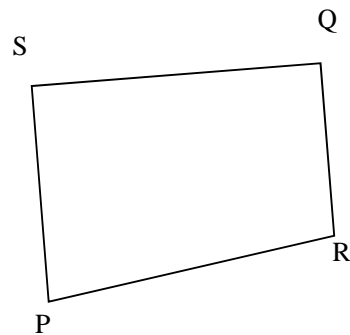


Figure 4 (b) quadrilateral SQR P

2. How else can you write $JKLM \equiv QSPR$ so that it remains true?



Feedback

We name the polygons by arranging letters whose properties are equal that is, **Angles** and **Sides**.

To answer question 1, the following must be observed.

$$\begin{aligned} \angle J &\equiv \angle R, \angle K \equiv \angle P, \angle L \equiv \\ \angle S &\text{ and } \angle M \equiv Q \end{aligned}$$

$$\overline{JK} \equiv \overline{PR}, \overline{LM} \equiv \overline{QS}, \overline{JM} \equiv \overline{QR} \text{ and } \overline{KL} \equiv \overline{PS}$$

So the quadrilateral JKLM is congruent to quadrilateral PRQS.

This can be presented using the congruent symbol as shown below;

$\Delta ABC \equiv \Delta ABD$ (the expression $\Delta ABC \equiv \Delta ABD$ means triangle

ABC is exactly the same as triangle ABD).

We can also use the angles to determine congruence in figure as:

$$\angle A = \angle A, \angle B = \angle B, \angle C = \angle D.$$

If all corresponding angles of the two triangles in Figure 1(a) and 1(b) are equal then this means that the two triangles are congruent. As already stated above the angle $A =$ angle A and angle $B =$ angle B and that angle $C =$ angle D .

We can apply the same reasoning used above to any other figures or shapes to determine whether these figures or shapes are congruent.

Let us consider how the above reasoning could apply to triangles.

In line with the above, we can now discuss the general rule for congruent in triangles as:

“Two triangles are congruent if and only if their corresponding angles and corresponding sides are equal”.

We can summarise congruency of any two triangles under the following four (4) conditions as:

Condition 1

If the 3 sides of one triangle are equal to the three corresponding sides of another triangle, then the two triangles are congruent.

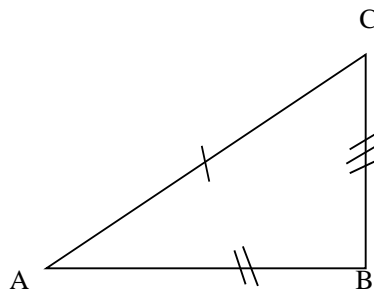


Figure 5 (a) triangle ABC

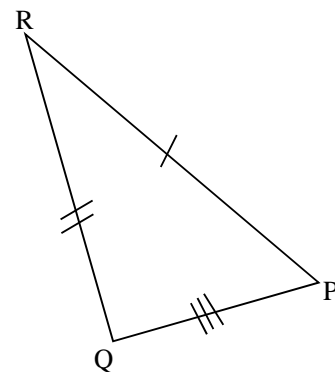


Figure 5 (b) triangle PQR

The above statement is stated in short by comparing the sides of the above triangles as follows:

$AC = PR$, $AB = RQ$ and $BC = QP$. ‘/’ means that side AC is equal to side RP , ‘//’ means that side AB is equal to side RQ and

'///' means that side BC is equal to side QP. When comparing two or more figures which have the corresponding sides equal we use a single '/' for first sides which have the same measurement, the double '// ' for the second set of equal corresponding sides, the third set will be represented by '///' for a third set of equal corresponding sides. If there was a fourth equal corresponding sides will have '////' and so on.

The corresponding sides of the two triangles can be expressed as $\overline{AC} \equiv \overline{PR}$, $\overline{AB} \equiv \overline{QR}$ and $\overline{BC} \equiv \overline{QP}$. This means that the sides are corresponding and the property is called the side, side, and side, abbreviated as SSS.

Hence, the first condition is called **SSS**.

If you identify that each of the corresponding sides of the triangles are equal, then the two triangles must be congruent.

In this case, $\triangle ABC \equiv \triangle PQR$ (\equiv , means congruency; a short way of writing congruent).

Condition 2

If two sides and the included angle of one triangle are equal to the corresponding two sides and the included angle of another triangle. When these conditions are fulfilled then the two triangles are congruent. The included angle is the angle between two arms that form an angle. You can see this in the drawing shown below.

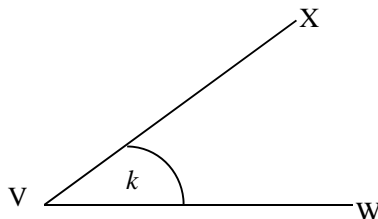


Figure 6 showing angle X VW.

In the above VX is the side and VW is also a side while X VW is the included angle. This property is called Side, Angle, and Side (**SAS**).

To consolidate the property of **SAS**, relate our discussions to figures 5(a) and 5(b) shown below.

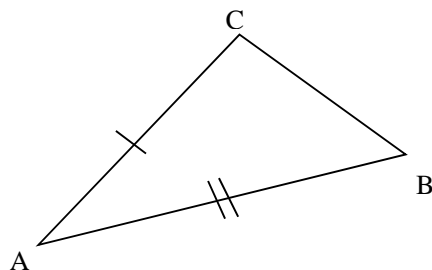


Figure 7 (a) triangle ABC

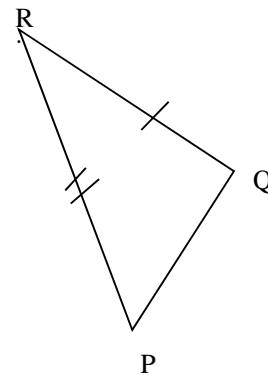


Figure 7 (b) triangle PQR

In the above triangles, you will discover that; $AB = PR$, $AC = RQ$ and $\angle BAC = \angle QRP$.

Please note that the angle used is the included angle between the two sides. When we check on the information given we find that we have been given the two corresponding sides that is AB is corresponding to PR and side AC is corresponding to side RQ.

Now, if we look at $\angle BAC$, means angle A, the letter which is in the middle of the three letters refers to the angle. This angle is the one referred to as the included angle. An angle is written in short form using the symbol “ \angle ” which simply means angle.

Therefore, $\triangle ABC \equiv \triangle PQR$. In this expression (\triangle stands for triangle and \equiv stands for exactly the same, thus $\triangle ABC \equiv \triangle PQR$ means triangle ABC is exactly the same as triangle PQR.

Or this can be presented as $\begin{matrix} A & B & C \\ P & Q & R \end{matrix}$ meaning that A is corresponding to P, B is corresponding to Q and C is corresponding to R. $\begin{matrix} A & B & C \\ P & Q & R \end{matrix}$, this is another way of expressing sides or angles which are congruent.

Condition 3

Two triangles are congruent if any two angles of the first triangle are equal to the corresponding two angles of the second and one side is equal to the corresponding side of the other.

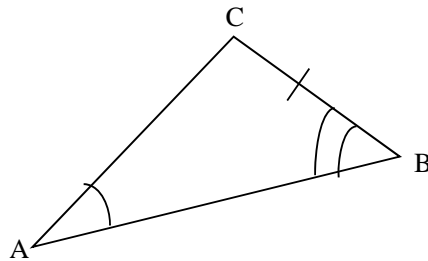


Figure 8(a) triangle ABC

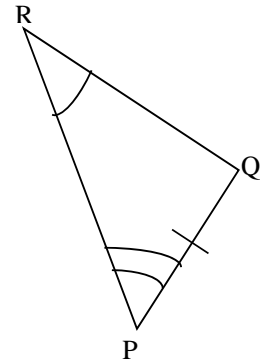


Figure 8(b) triangle PQR

In the above triangles, we note the following:

$$\angle BAC = \angle PRQ, \angle ABC = \angle RPQ \text{ and } BC = PQ.$$

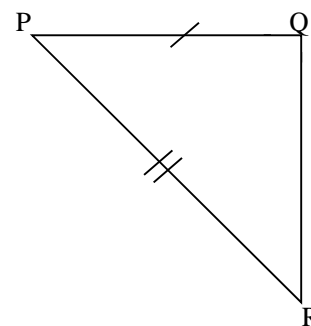
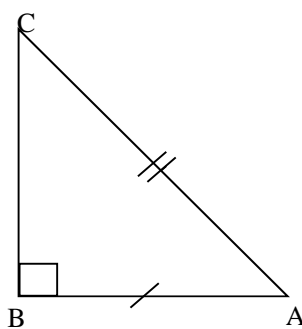
This property is called Angle, Angle Side [AAS].

Therefore,

$$\triangle ABC \equiv \triangle PQR$$

Condition 4

If two right angled triangles have the hypotenuse (hypotenuse is the longest side of a right angled triangle) and another side of the one triangle equal to the corresponding hypotenuse and another side of the other triangle, then the two triangles are congruent.



*Figure 9 (a) triangle ABC**Figure 9 (b) triangle PQR*

In the above triangles, we should note the following:

$$\angle B = \angle Q = 90^\circ, AB = PQ \text{ and } AC = PR$$

This property is called Right angle, Hypotenuse, Side [RHS]

Therefore, $\triangle ABC \equiv \triangle PQR$

We have now completed the four conditions for congruency using the triangles. But it is important to note that congruency can apply to other figures as well. This would mean that such figures would have the shapes, sizes, angles and areas the same or equal.

For example circles of same area are congruent. Also squares of the same area are congruent.

In conclusion, congruent shapes are shapes which are exactly the same.

We have just concluded discussing the four conditions which satisfy congruency in figures. These conditions are the bases for determining whether the figures being compared are congruent figures. These four conditions are important because they will help us identify corresponding sides and angles which are congruent in figures.

Let us now move on to the next section, where we will explore how to solve problems involving congruency. You will realise that the discussions in next section will be addressing the second unit outcome.

Using congruency in solving problems

Let us consider the four conditions discussed in solving problems involving congruency. This example is meant to build the understanding that in congruency we aim at proving the four conditions if they are fulfilled.

Example 1

Solving problems requiring simple logical deduction

Consider the figure given below.

AXB, BXD are straight lines. $AX = DX, BX = CX$, prove that
 Angle $A =$ Angle D

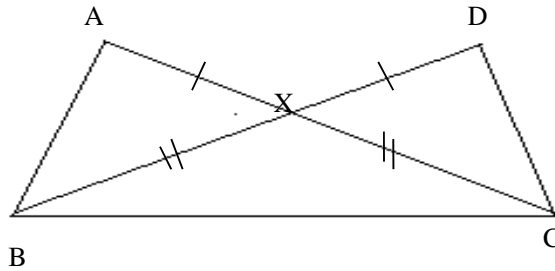


Figure 10 showing triangles ABC and BCD

Solution

$BX = CX$ given

$AX = DX$ given

Angle $AXB =$ Angle DXC vertically opposite angles

Therefore, **Angle $A =$ Angle D SAS**

Let us consider another example.

Example 2

Supposing you are given the following information;

That is $\angle EFD$ and $\angle EFG$ are right angles.

$$\angle DEF \equiv \angle GEF$$

Prove that $\overline{DF} \equiv \overline{GF}$

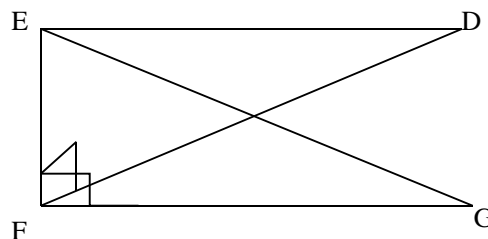


Figure 11 showing two congruent triangles.

Solution

First, prove that, $\triangle DEF \equiv \triangle EFG$. Then, $\overline{DF} \equiv \overline{GF}$, since they are corresponding parts of congruent triangles.

This can be presented using the table below.

<i>Statement</i>	<i>Reason</i>
EFD and \angle EFG are right angles	<i>Given</i>
2. $\angle EFD \equiv \angle EFG$	<i>All right angles are congruent</i>
3. $\overline{EF} \equiv \overline{EF}$	<i>Reflective property</i>
4. $\angle DEF \equiv \angle GEF$	<i>Given</i>
5. $\triangle EFD \equiv \triangle EFG$.	<i>ASA(angle side angle).</i>
6. $\overline{DF} \equiv \overline{GF}$	<i>Definition of congruent triangles</i>

Let us discuss the last example to consolidate the skills, knowledge and concept of congruence on solving problems involving congruence.

Example 3

Prove that $JM \parallel IK$. Given that $\overline{JK} \equiv \overline{LM}$; $\overline{JM} \equiv \overline{LK}$.

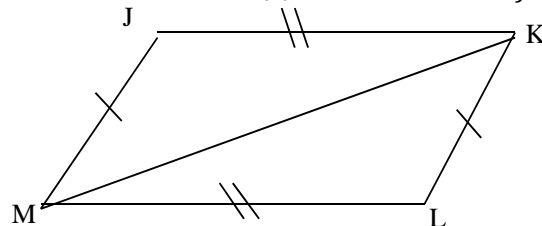


Figure 12 showing a parallelogram**Solution**

<i>Statement</i>	<i>Reason</i>
$\overline{JK} \equiv \overline{LM}; \overline{JM} \equiv \overline{LK}$	<i>Given</i>
2. $\overline{KM} \equiv \overline{MK}$	<i>Reflective property</i>
3. $\triangle JKM \equiv \triangle LMK$	<i>SSS</i>
4 $\angle JMK \equiv \angle LKM$	<i>Definition of congruent</i>
5. $\overline{JM} \equiv \overline{LM}$	<i>If two lines are intersected by a transversal and alternate interior angles are \equiv then the lines are //</i>

As we come to the conclusion of this section let us review what we have discussed. So far we have used the four conditions for congruency to solve and prove congruence in figures. We have also given reasons for congruence in figures based on the four conditions discussed earlier in Section 2.

Let us now move on to the next section, where we will explore congruency in the natural environment. You will realise that the discussions in next section will be addressing the third unit outcome.

Identifying congruency in the natural environment

In this section, we aim at achieving the third outcome. In this section we are going to learn how to identify things around us to the understanding of mathematics. This will also help us apply the knowledge and skills to solve problems relating to our everyday situations.

In our environment there are so many things that look exactly the same. These are usually buildings, plants and animals. There are

certain things that just grow to a certain height and size. Such things are referred to as congruent.

In animals we have such animals as the bees, ants, termites, grasshoppers, insects and many others. To a certain extent we have even people who look exactly the same these are usually identical twins.

The knowledge of congruency is widely used in our day to day situations. The knowledge is usually applied in construction industries and manufacturing industries such as textiles industries.

The content in this section is not new since it has been dealt with in the previous Unit, 8. adequately.

Let us go through an example to consolidate the understanding of congruent.

Example 4

What steps would you take, if a glass pane which is in a form of regular pentagon measuring 60cm is broken and need replacement?

Solution

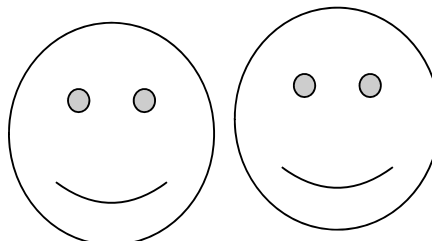
You would need to take the measurement so that the glass pane is cut out to match the original exactly.

Let us consider other examples about the natural environment. This is meant to drive you into the understanding that congruent figures are mostly used in designs (fashion), building industries and manufacturing industries just to mention a few.

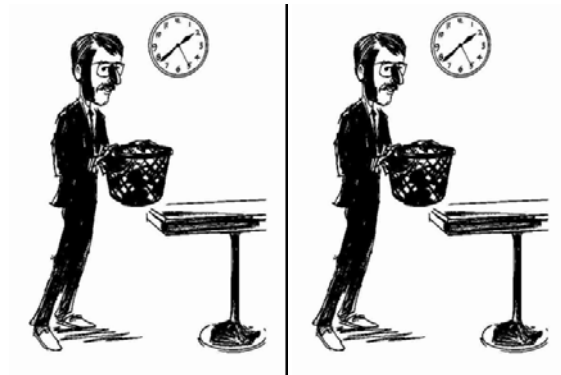
When a car or machine has a breakdown in terms of a worn out part and need replacement one has to find the exactly part that will fit to replace the worn out part. In building or construction of houses you will find that certain houses are built exactly of the same size and everything about those houses are exactly the same. If it happens that they are the same shape and same size then we can conclude that the houses are congruent.

To consolidate the understanding of congruence you can compare the following pairs of figures.

(i)



(ii)



(iii)



Figure 13: 3 pairs of congruent figures

What was your conclusion about these figures? I am sure it was that they are congruent.

With this understanding you can now do some activity. This activity is meant to help you apply the knowledge, skills and concept of congruency.

Activity 6



Activity6

Draw three pairs of objects that are congruent. Use the space provided below for your drawings.



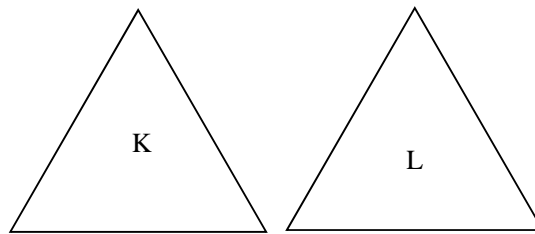
Feedback

Feedback

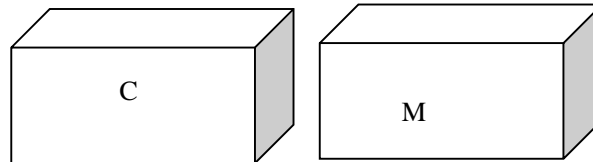
In order to have objects that are congruent, it is good to use the measuring instruments to help us draw objects that will have the same size and shape.

In my case, I will use the following shapes triangles, cuboids and cylinders.

(i)



(ii)



(iii)

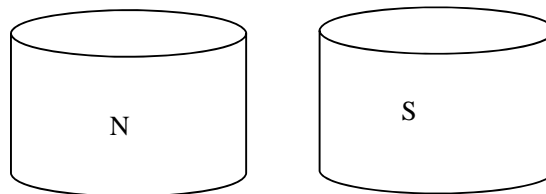


Figure 14: Different shapes-triangles, cuboids & cylinders

Your answer does not have to be the same as mine. What is important is that the shapes must be congruent.

We have now come to the end of Topic 1. You can now turn to the assignment section of this unit where you will find the end of topic exercise number one, which will help you to assess how well you have understood the topic.

You should only check the answers after you have gone through all the questions.

This marks the end of our discussion concerning congruency, you can now read the topic summary.



Topic Summary

In this topic, you learnt how to identify congruency in figures by comparing corresponding angles and sides. You further learnt how to prove congruency by solving problems requiring simple logical deductions in congruency. In addition you learnt how to use the four conditions to test for congruence. Finally, you were guided towards understanding congruence in our environment by citing examples of different congruent figures.

So far we have managed to achieve the first three learning outcomes stated in the introduction.

In the next topic, you will learn how find the ratios of sides of similar figures. You will also learn how to calculate the unknown sides or angles in similar figures and finally you will learn how to identify similarity in the natural environment.

I trust that you have completed the end of topic exercise and that you are happy with your progress. If you have not yet done the exercise, do so now before starting the next topic.

Topic 2: Similarity

In the previous topic we concentrated on congruence. You learnt how to identify congruency in figures and other shapes. You further learnt how to use congruency in solving problems requiring simple logical deductions. Finally, you learnt how to identify congruency in the natural environment.

In this Topic 2, we are going to concentrate on the concept of similarity. In order to achieve this, we will discuss how to find the ratios of sides of similar figures. You will also learn how to calculate the unknown sides or angles in similar figures and finally you will learn how to identify similarity in the natural environment.

This topic is divided into four sections which are meant to address the unit outcomes as stated below.

Therefore, by the end of our discussion of the topic, you are expected to achieve the following outcomes:



Outcomes

- *Find* the ratio of sides of similar figures.
- *Calculate* the unknown sides or angles in similar figures.
- *Calculate* the area and volumes of similar figures.
- *Identify* similarity in the natural environment

To achieve these outcomes, the following objectives should be achieved by the end of this topic. You should be able to:

- Identify corresponding sides in figures.
- Write sides in ratio form
- Compare sides and angles in any given figure

Now, let us begin our discussion of the first section by considering the first topic outcome.

Finding the ratio of similar figures

You will remember that in the first section of Topic 1, we discussed the difference between congruence and similarity. We highlighted that although similar figures may appear the same, they could be different in size. Now let us explore this notion further.

Imagine that Sipeso and Mutale are pen-pals. Sipeso stays in Lusaka while Mutale lives in Luwingu. Sipeso invites Mutale for a holiday and reminds Mutale to carry her [Sipeso's] photograph for

identification. Sipeso looks the same as the person in the photograph.

You will notice that Sipeso's photograph shows a figure which has the same shape as Sipeso; that figure is much smaller than the real Sipeso. This is a very good example of similar figures. This demonstrates that similar figures may have the same shape but they do not necessarily have the same size. This leads us to our first objective which is about identifying corresponding sides in similar figures.

Identify corresponding sides in figures

Let us consider the rectangles below to identify whether the rectangles are similar.

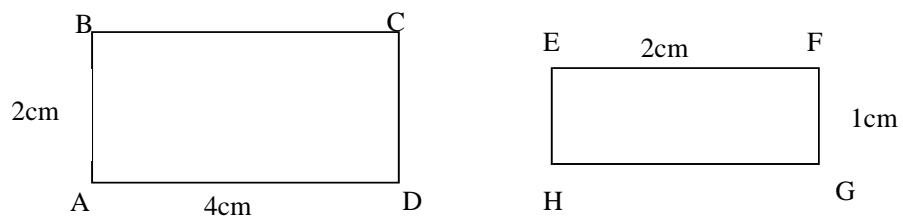


Figure 15 (a) shows rectangle ABCD Figure 15(b) shows rectangle EFGH

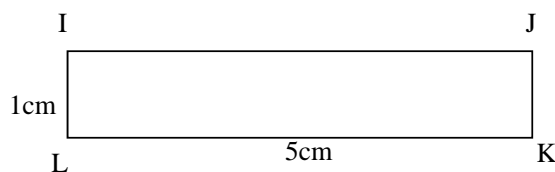


Figure 15(c) shows rectangle IJKL

The figures above are three rectangles of different sizes. Let us use these rectangles to identify corresponding sides. Since these are rectangles their lengths are corresponding and their widths are also corresponding. Based on this explanation you will clearly notice that side AB is corresponding to sides EH and IL. In the same way you will notice that side BC is corresponding to EF and IJ.

The three figures are all rectangles but not all of them are similar.

This can be proved by the use of ratios.

Writing sides in ratio form

Comparing the corresponding sides of figures 15(a) and 15(b) above in terms of numbers we have $AB : EH = 2 : 1$, $BC : EF = 4 : 2$, $CD : FG = 2 : 1$ and $DA : GH = 4 : 2$. Figure 15(a) and 15(b) are

similar because all the corresponding sides have the common ratio, which is 2 : 1

Now, let us compare the corresponding sides of figure 15(a) and figure 15(c) to find out if the two figures are similar or not. The following are corresponding sides; $AB : LI = 2 : 1$, $BC : IJ = 4 : 5$, $CD : JK = 2 : 1$ and $DA : KL = 4 : 5$. Since all the three sides of the two figures have no common ratio then the two figures are not similar.

Now do this activity to put into practice what you have just learnt.



Activity

Activity 7

Show whether figure 15(b) and figure 15(c) are similar or not using ratios



Feedback

The ratios of corresponding sides of figure 15(b) and figure 15(c) are;

$EF : IJ = 2 : 5$, $FG : JK = 1 : 1$, $GH : KL = 2 : 5$ and $HE : LI = 1 : 1$.

Figure (b) and figure (c) are not similar, since all the corresponding sides have no common ratio.

We have just seen how a common ratio is used to determine similarity in figures. Let us move on to deal with how sides and angles are also used to determine similarity.

Comparing sides in any given figure

Let us consider the two triangles in the example below.

Example 4

$\triangle ABC \sim \triangle DEF$, this expression means that triangle ABC is similar to triangle DEF. The symbol ‘ \sim ’ means similar and the symbol ‘ \triangle ’ means triangle.

Show that the lengths of corresponding sides are proportional.

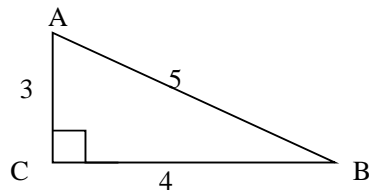


Figure 16a: Triangle ABC

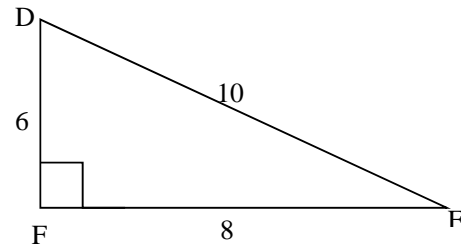


Figure 16b: Triangle DEF

Solution

First we begin by calculating the ratios of lengths of corresponding sides.

$$\text{That is; } \frac{AB}{DE} = \frac{5}{10} = \frac{1}{2}, \frac{BC}{EF} = \frac{4}{8} = \frac{1}{2} \text{ and } \frac{CA}{FD} = \frac{3}{6} = \frac{1}{2}$$

Since all the three ratios are the same. The lengths of corresponding sides are proportional. $\frac{1}{2}$ or 1 : 2 is called the common ratio. This will make us conclude that the two figures are similar but if the three ratios were not the same then the two figures will be termed to be Not similar.

Let us consider another example where we are going to compare angles in figures to enable us to identify similarity in figures.

Comparing angles in any given figure

To understand how similarity can be proved by comparing angles here are conditions that have to be fulfilled.

These are:

- Two figures are said to be similar if the angles of the first figure are equal to the angles of the second figure, and their corresponding sides are proportional as can be seen in Figures 16(a) and (b).

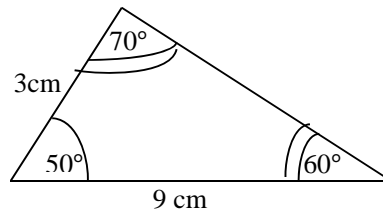


Figure 15 (a): Comparing angles

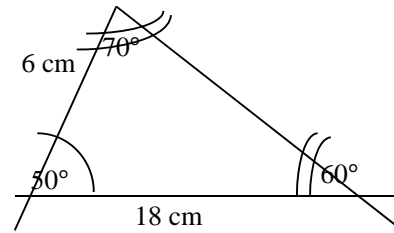


Figure 15(b): Comparing angles

In conclusion, two figures are similar if their corresponding angles and their corresponding sides are in the same ratio.

Now, do Activity 8 below to consolidate on what we have just discussed. This is also meant to help you assess yourself on how much you have learnt.

Activity 8



ctivity

1. Show whether the pairs of figures shown are similar or not similar

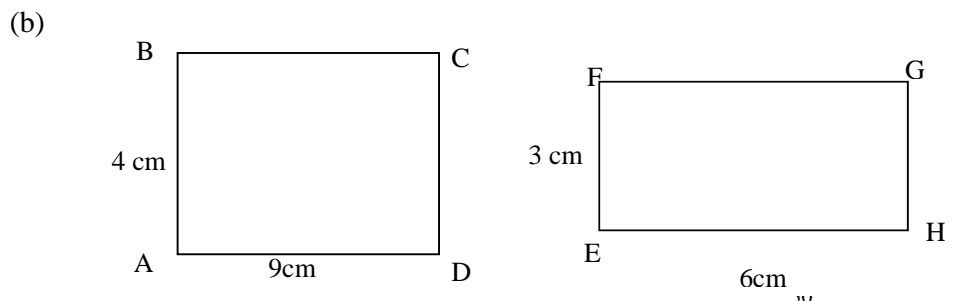
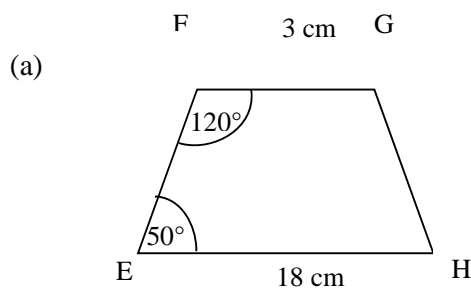
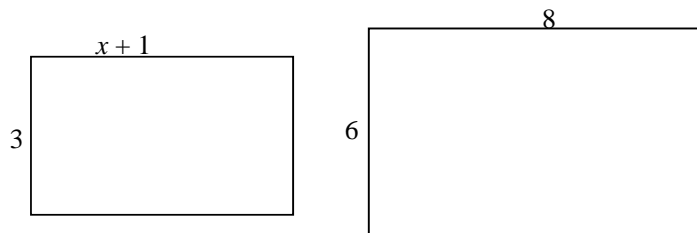


Figure 16: Activity 8 figures on similarity

2. The two polygons are similar. Solve for x .

(a)



(a)

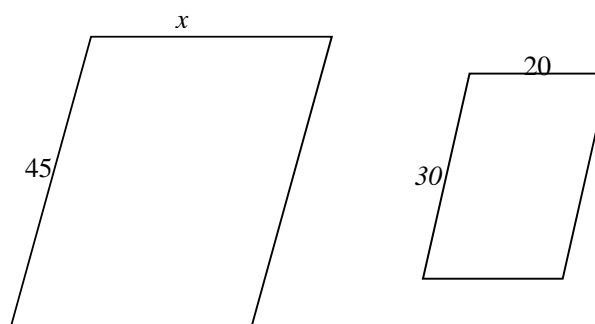


Figure 17: Activity 8 polygons for solving similar



Feedback

Question 1

(a) $\frac{FG}{BC} = \frac{3}{1}$, $\frac{EH}{AD} = \frac{18}{6} = \frac{3}{1}$. Therefore, the two figures are similar, since their corresponding sides have the common ratio, that is 3 : 1.

(b) Side AB is corresponding to side EF and are in the ratio 4: 3 while side AD is corresponding to side EH and are in the ratio 9: 6. Since the two corresponding sides have no common ratio then the two figures are not similar. These ratios are 4:3 and 9: 6.

Question 2

(a) To solve for x we are first going to represent the corresponding sides as ratios as shown below.

$$\frac{x+1}{8} = \frac{3}{6}, \text{ we cross multiply this expression.}$$

$6(x + 1) = 8 \times 3$, we going to remove the brackets by multiplying are each in $(a + 1)$ by 6.

$$6x + 6 = 24$$

$6x = 24 - 6$, collecting like terms.

$$6x = 18$$

$\frac{6x}{6} = \frac{18}{6}$, dividing both sides by 6 we are going to get the value for x .

Therefore, $x = 3$.

(b) To solve for x we are first going to represent the corresponding sides as ratios as shown below.

$$\frac{x}{20} = \frac{45}{30}, \text{ we cross multiply this expression.}$$

$$30x = 45 \times 20$$

$\frac{30x}{30} = \frac{45 \times 20}{30}$, we divide both sides by 30 so that we remain with x .

$$x = 30$$

Therefore, x is 30.

We hope you enjoyed working through this section. Let us proceed to the next section where will discuss how to calculate the unknown sides or angles in similar figures. To facilitate easier

understanding let us relate this discussion to the study of conditions that you learnt in the previous section.

Calculating the Unknown Side or Angles in Similar Figures

This section will address outcome number two of this topic; which requires us to discuss how to calculate the unknown sides or angles in similar figures. For purposes of our discussion we shall use the triangle to amplify our case, using some of the information discussed earlier in this topic. For example in the above section we investigated a number of conditions which can assist us to prove similarity. Let us explore how all these situations are further based on the following three conditions for all figures or triangles:

Condition 1

Two triangles or figures are similar if their corresponding angles are equal. This property is also called equiangular case i.e. Angle, Angle, Angle [AAA].

Let us consider figures below.

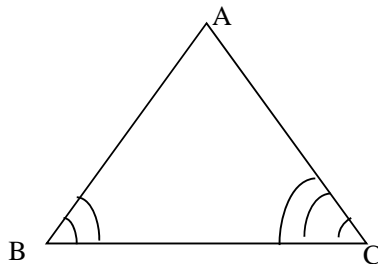


Figure 18(a): Unknown angles

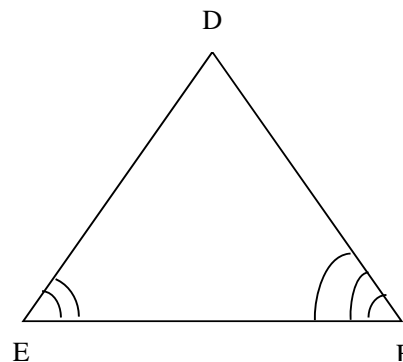


Figure 18(b): Unknown angles

Here, $\angle A = \angle D$ $\angle B = \angle E$ and $\angle C = \angle F$

The two triangles are equiangular if they are similar.

i.e. ΔABC is similar to ΔDEF

You should remember to arrange the letters in similar figures

Condition 2

Two given triangles are similar if their corresponding sides are proportional. Let us consider figures below.

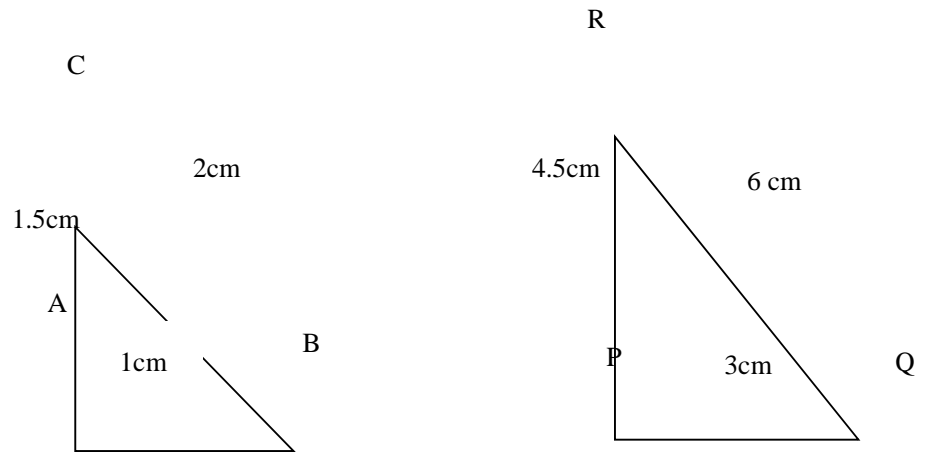


Figure 20(a) & (b): Corresponding sides of angles

From the figures above, we see the ratios:

$$\frac{AB}{PQ} = \frac{1}{3} \qquad \frac{AC}{PR} = \frac{1.5}{4.5} \qquad \frac{BC}{QR} = \frac{2}{6}$$

Since these ratios are equal the sides are proportional, it follows that the triangles are similar.

i.e. ΔABC is similar to ΔPQR .

Condition 3

Two triangles are similar if two pairs of their corresponding sides are proportional and the included angles are equal.

Let us consider the figures below:

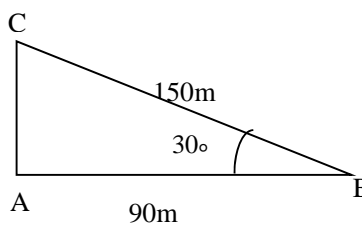


Figure 21 (a)

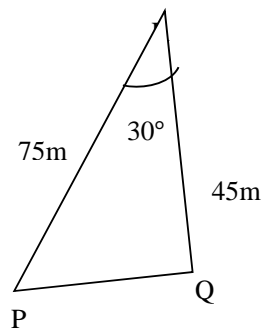


Figure 21 (b)

Figures 21a & b: Corresponding sides and included angles

From the figures above,

$$\frac{AB}{QR} = \frac{90}{45} \quad \frac{BC}{PR} = \frac{150}{75} \quad \text{and the included angle is the same.}$$

Therefore, the two triangles are similar

i.e. $\triangle ABC$ is similar to $\triangle PQR$

Let us consider an example to appreciate what we have learnt so far.

Example 5

In the triangle ABC, PQ is parallel to BC, $AB = 3AP$ and $BC = 6\text{cm}$.

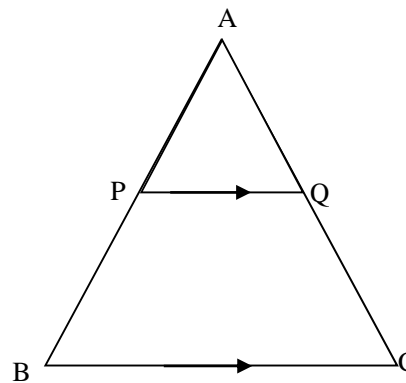


Figure 22 showing two triangles in one.

Find the length of PQ.

Solution:

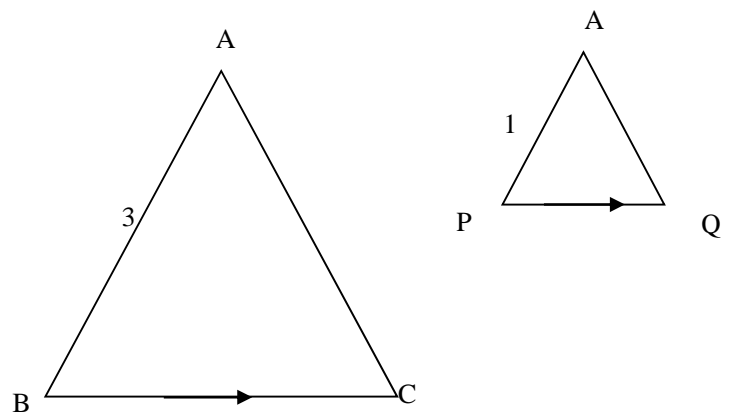


Figure 22(a): Solution on two triangles in one example

The diagram has been split into two similar triangles as shown above. So we are going to use the two similar triangles to solve the problem for PQ.

Angle APQ and angle ABC are equal and corresponds to each other.

Side AP correspond to side AB. Using the ratio of these two sides, we get:

$$\frac{AP}{AB} = \frac{1}{3}. \text{ This was got from the statement given in the question. The statement was: } AB = 3AP.$$

This statement means 3 of AP lengths makes up 1 AB length.

So to find the length of PQ, we are to use the ratio: $\frac{AP}{AB} = \frac{1}{3}$

Using ratios of sides, we have the following:

$$\frac{AP}{AB} = \frac{PQ}{BC} = \frac{AQ}{AC} = \frac{1}{3}$$

Since we are looking for PQ, we are going to use the first part as follows: $\frac{AP}{AB} = \frac{PQ}{BC}$

Then we substitute AP with 1, AB with 3 and BC with 6;

$$\frac{1}{3} = \frac{PQ}{6} \text{ (cross multiply)}$$

$$PQ \times 3 = 6 \times 1$$

$$3PQ = 6 \text{ (divide by 3 both sides)}$$

$$PQ = 2.$$

Therefore, the length of PQ is 2 cm.

We have just finished our discussion on how to find the length of the missing or unknown side. Now let us discuss how to find the missing or unknown angle of similar figures.

Two angles are similar if they have the degree measurement. Now, let us consider the example below:

Example 6

The triangles in this pair are similar. Find the value of x .

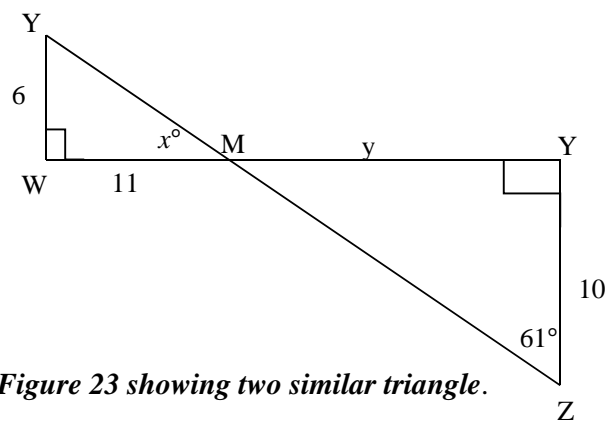


Figure 23 showing two similar triangle.

Solution

Statement	Reason
$\angle MYZ \equiv \angle YWM$	Given
$\angle YZM \equiv \angle WYM = 61^\circ$	Given

Therefore, angle $x = 180^\circ - (90^\circ + 61^\circ)$.

This will give us $x = 29^\circ$

So far, we have managed to deal with the first and second topic outcomes. We are now moving on to discussion on how to calculate the area and volume of similar figures.

Calculating the Area and Volume of Similar Figures

This section is aimed at addressing the third outcome stated for this topic. You will realise that in this outcome we have to deliberate on two specific aspects regarding similar figures. The first part of the outcome requires us to discuss how to calculate the area of similar figures. The second part requires us to discuss how to calculate the volume of similar figures. Let us start by discussing how to calculate the area of similar figures.

Calculating the Area of Similar Figures

Here we want to find the ratios of areas of similar figures. We consider the ratios of areas of the figures below.

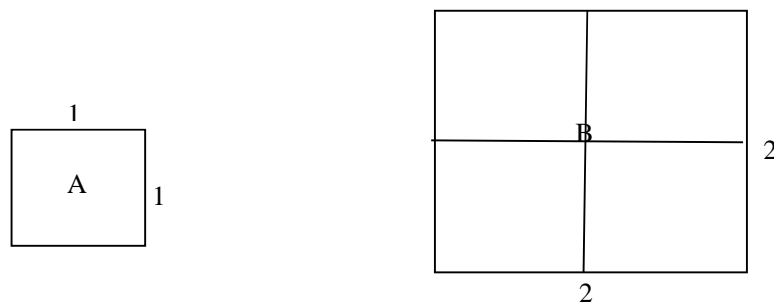


Figure 24 (a): Calculating area

Figure 24 (b): Calculating area

The area of A is 1 square units and the area of B is 4 square units.

If we write these areas as ratios, we shall get something like this:



Note it

Now remember how to find the area of the square. The area of a square is found by multiplying length by length $A = l \times l$

Therefore, Area A: Area B is equals to $1 \times 1 : 2 \times 2$

Which is, $1^2 : 2^2$

Thus, $1 : 4$

Writing the ratio of the sides of the same figures, we get:

Side of A: Side of B

$$1 : 2$$

Comparing the ratios, we find that if the ratio of sides is $m:n$, then the ratio of their areas is $m^2:n^2$.

Now we know that when the number 1 is squared the solution is 1. This means that 1 is raised to the power 2 (squared means 1 by 1 is one).

Now let us deal with another situation of a similar kind.

Consider the two figures given below to find the areas using ratios.

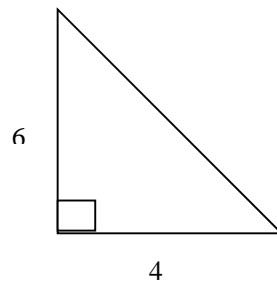


Figure 25(a): Calculating area 2

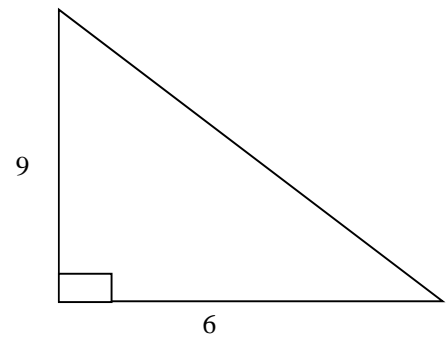


Figure 25(b): Calculating area 2

The ratios of sides are: 4: 6 and 6: 9. We are to use one ratio and reduce it to the lowest terms.

Using the ratios 4 : 6,

$$\begin{aligned} 4 : 6 &= \frac{4}{2} : \frac{6}{2} \text{ (Divide both sides by 2)} \\ &= 2:3 \end{aligned}$$

Therefore, $4 : 6 = 2 : 3$.

The ratio of areas is calculated by using the ratio of sides and squaring the terms.

Ratio of areas:

$$\begin{aligned} &2^2 : 3^2 \\ &= 4 : 9 \end{aligned}$$

Therefore, the ratio of areas is 4 : 9.

We use the ratio concept to solve problems involving areas of similar figures. We are going to consider the following example to learn how to use the ratio to calculate the area of a triangle.

Now let us consider another different situation. Here we are to deal with calculating areas of triangles using similar figures ratio idea.

Example 7

The given triangles are similar. Find the area of the larger triangle, given that the area of the smaller triangle is 16 cm^2 .

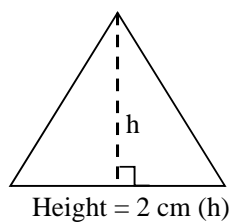


Figure 26a: Area of similar triangles

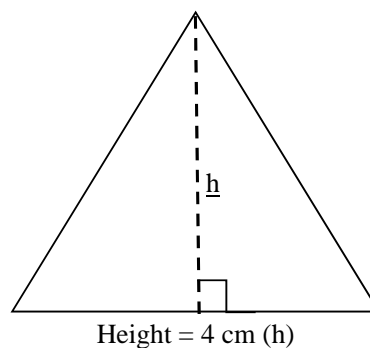


Figure 26b: Area of similar triangles

Solution

$$\begin{aligned} \text{Ratio of sides} &= 2:4 \\ &= 1:2 \end{aligned}$$

Therefore, Ratio of areas = 1:4

The ratio of areas is calculated by using the ratio of sides and squaring the terms.

Ratio of areas:

$$\begin{aligned} &= 1^2:2^2 \\ &= 1:4 \end{aligned}$$

Therefore, the ratio of areas is 1: 4

Now let x be the area of the larger triangle. That is 1: 4.

$$1: 4 = 16: x$$

$$\begin{aligned} \text{Therefore, } \frac{1}{4} &= \frac{16}{x} \\ x &= 4 \times 16 \\ x &= 64 \end{aligned}$$

Therefore, area of the bigger triangle = 64cm^2 .

This brings us to the end of the discussion of the first part of this section. In this part we discussed the first part as shown in the title of the stated outcome i.e. how to find areas of similar figures. We have discussed how areas in similar figures are calculated using their corresponding ratios of their sides.

Now let us proceed with the discussion of the second part of this section; how to find volumes of similar figures

Calculating the Volumes of Similar Figures



Note it

If the ratio of the sides of two similar figures is $a : b$, then the ratio of their areas is $a^2 : b^2$. Equally their volumes will $a^3 : b^3$.

Basically, to find areas of similar figures we were using ratios of corresponding sides of the two figures. Now, let us use the same concept and knowledge used in finding areas of similar figures to find the volumes of similar figures.

To find the areas of two similar figures we were squaring the ratios of the two figures equally to find the volumes of two similar figures.

We cube the ratios or in other words we raise the ratio to power three for volume.

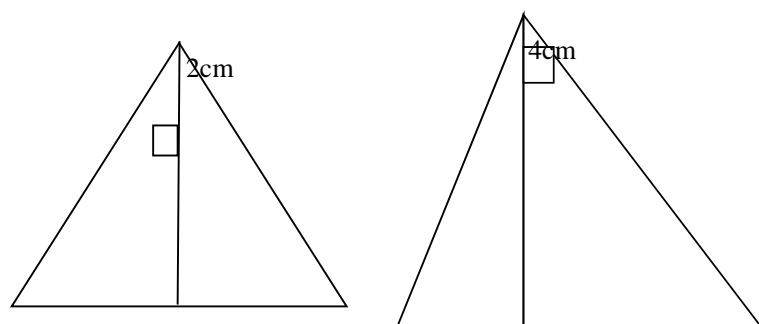
Now, do the activity below to build on the skills and knowledge gained in this section. This is meant to allow you to have more practice on how to handle question relating to areas and volumes of similar figures.

Activity 9



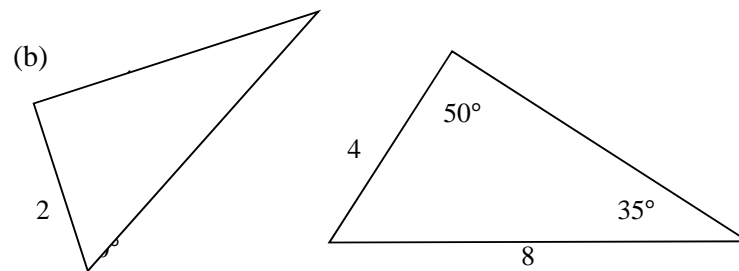
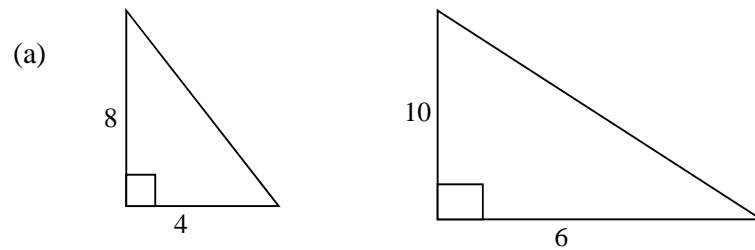
Activity3

1. The given triangles are similar. Find the area of the larger triangle, if the area of the smaller triangle is 16cm^2



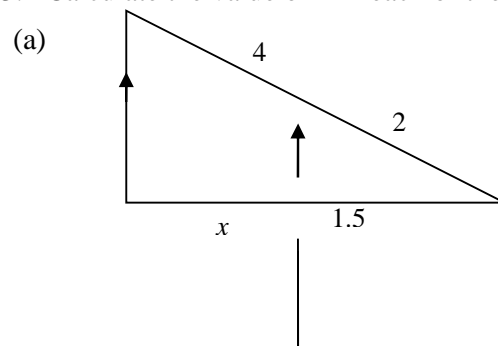
Figures 27a & b: Finding area of triangles

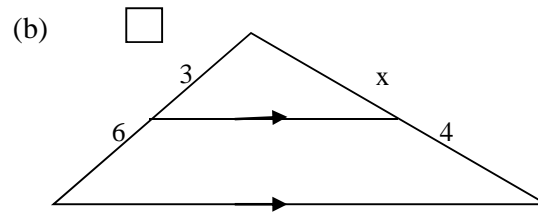
2. This question is meant to help you apply the concept of comparison in relation to ratios of the sides of the figures given. This activity is also meant to assess your understanding of this section. State whether the following pairs of triangles are similar or not.



Figures 28 a & b: Comparing similar triangles

3. Calculate the value of x in each of the following triangles





Figures 29a & b: Calculating the value of x in triangles

- The sides of a triangle are 6cm, 8cm and 9cm. The shortest side of a similar triangle is 3cm. Find the lengths of the other two sides and the ratio of the areas of the two triangles.
- Two similar spheres have radii in the ratio 2 : 1. If the volume of the smaller sphere is 20cm^3 . Find the volume of the larger sphere.
- Two similar blocks have corresponding edges of length 15cm and 25cm. Find the ratio of their masses.



Feedback

Smaller Larger

1. Ratio 2 : 4

Area: 16 : x

$$2x = 16 \times 4$$

$$x = 8 \times 4$$

$$\mathbf{x = 32 \text{ cm}^2}$$

2. (a) **Smaller** **Larger**

$$\text{Ratio of sides: } 8 : 10 = 4 : 5$$

$$4 : 6 = 2 : 3$$

The two triangles are **Not similar**, since their corresponding sides are not giving us the same ratio when reduced in their lowest term.

(b) Not similar, because their corresponding sides are not in the same ratio.

(c) Not similar

3. (a) $\frac{2}{4} = \frac{1.5}{x}$

$$2x = 4 \times 1.5$$

$$x = 2 \times 1.5$$

$$x = 3$$

(b) $\frac{3}{6} = \frac{x}{4}$

$$6x = 3 \times 4$$

$$x = 1 \times 2$$

$$x = 2$$

4. We are given that the lengths of the larger triangle are; 6: 8: 9.

Let the lengths of the smaller triangle be 3: x: y.

By comparing corresponding sides we have 6: 3, 8 : x and 9 : y.

This is also expressed as; $\frac{6}{2} : \frac{8}{x} : \frac{9}{y}$.

Thus, the length of the unknown sides of the smaller triangle can be calculated as shown,

$$\frac{6}{3} = \frac{8}{x} \text{ (cross multiplying).}$$

$$\frac{6x}{6} = \frac{24}{6}$$

Dividing both sides by 6

$$x = 4$$

$$\frac{6}{3} = \frac{9}{y}$$

Cross multiplying

$$6y = 9 \times 3$$

$$\frac{6y}{6} = \frac{27}{6}$$

Dividing both sides by 6

$$y = 4.5$$

Therefore, the lengths of the smaller triangle will be for 4cm and for y will be 4.5 cm.

5. Pairing the corresponding sides with the corresponding volume we have; 2 : x and 1 : 20. Where x and 20 are representing the volume while 2 and 1 are representing the ratios of the two similar spheres

$$\frac{2}{x} = \frac{1}{20}$$

Therefore, $x = 40\text{cm}^3$.

Thus, the volume of the larger sphere is 40cm^3

We have now come to the end of the section where we were discussing how to calculate the area and the volume of similar figures. You will remember that earlier in topic one we discussed the issue of congruence in natural environment. Now in the next section, which is the final section of this topic, we shall investigate and give examples of how similarity can be identified in the natural environment.

Identifying Similarity in the Natural Environment

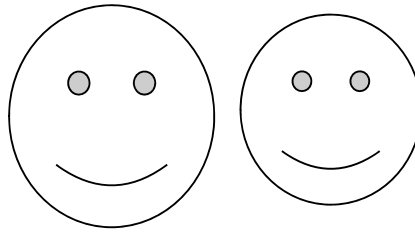
Similarity in our environment can be identified by simply looking at objects or things that appear the same but are or different sizes. A good example is a model of an object which is compared to the actual object.

This can be best understood by comparing the toys to the actual object such as a toy bicycle to the original bicycle. Children like playing around making imitations of the original things they see around them. All the things they make are similar to what is the actual object in the environment.

Another good example of similarity is seen in the construction, design and manufacturing industries. Now, let take an example in the building industry. You may have seen some buildings which have the same design but have different sizes. If you have seen such builds or such objects or thing then you can conclude that they are similar.

The following pairs of figures, some of which were introduced to you earlier in this unit, are similar:

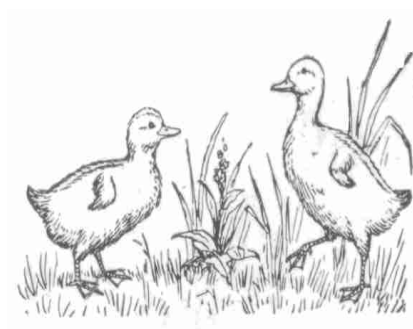
(i)



(ii)



(iii)



By this time you will have noticed that the above illustrations show figures which appear to be the same but have different sizes. This is in line with the definition which we discussed earlier at the beginning of topic one, where we indicated that similarity refers to a situation where two or more figures or shapes may have corresponding sides or angles in the same ratio. We also highlighted that such figures may appear the same but could be different in size. You do realise that this applies appropriately to the figures shown in the pictures above.

In conclusion, similarity involves comparison of things or objects which appear the same but are of different sizes. This discussion will help you in everyday life to easily identify similar figures. Finally, you can refer to the above section if you think you need more information about this section where similarity has been adequately covered.

So far, we have managed to deal with the last outcome of this topic which is also the final of the unit. This therefore brings us to discussion of the end of this unit. Before you proceed to the topic summary, we would like to invite you to turn to the end of this unit where you will find the end of Topic Exercise 3; which should assist you to assess yourself; on how well you have understood the topic. Remember that you should first work on the exercise and only check the answers thereafter.

You can now read a topic summary



Topic Summary

Topic 2 of unit 9 covered issues relating to similarity and dealt with four topic outcomes. These included the following: (i) how to find the ratio of similar figures. Here we learnt that when using ratios we simply square the ratios and what we will find will be the same answers to the ones that uses the formula ($A = L \times B$) for area and $V = L \times B \times H$ for volume). (ii) Further, we discussed how to calculate the unknown sides or angles in similar figures. (iii) In Section 3 we discussed how to calculate the area and the volume of similar figures using ratios. (iv) Lastly we discussed how to identify similarity in the environment. To help us achieve our objectives we showed you some examples on how to calculate areas and volumes of similar figures. You also did activities to help you understand the topic.

You were also encouraged to do the end of topic exercises which can be found at the end of the unit. We highlighted that it was very important that you first answer the questions in the exercises and then compare your solutions to those in the feedback. If you found some parts of the exercise rather difficult please revise the relevant sections of the topic.

Now you can proceed to the unit summary.



Unit Summary

You have come to the end of this unit. In this unit, we have covered congruency and similarity in different figures such as triangles and similarities of figures. We hope you have realised the learning outcome stated at the beginning of the unit.

The Unit had a total of seven unit outcomes and it was divided into two Topics. Topic 1 covered the first three unit outcomes and we started by discussing how to identify congruence in figures. You further learnt how to use congruence in solving problems requiring simple logical deductions. Then you also learnt how to identify congruency in the natural environment.

You should remember that Topic 2 covered the next four unit outcomes. These included: how to *find* the ratio of sides of similar figures; how to calculate the unknown sides of similar figures; how to calculate the areas and volumes of similar figures using ratios of similar and how to identify similarity in the natural environment. You also worked on a number of activities, and also did end of topic exercises.

This marks the end of Unit 9; you can now do the Tutor Marked Assessment -TMA (3) which cover questions from Units 7, 8 and 9. Find attached in the assessment section of the unit the TMA which you are expected to do before you move on to the next unit.

Make sure that you have attempted all the activities and you have submitted the TMA. This is meant to prepare you for final examinations ahead of your secondary programme.

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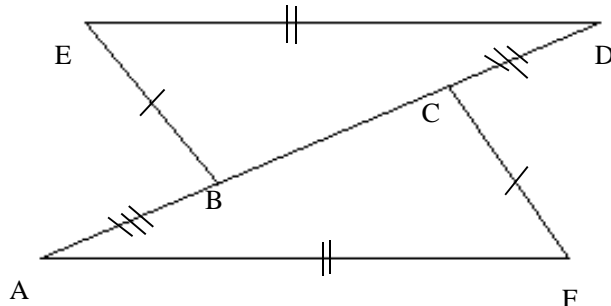
End of Topic Exercises



Assignment

Exercise 1

Prove that triangle ACF is congruent to triangle BDE.



If $AB = CD$ and $ABCD$ is a straight line, use the conditions for congruence to prove that ED is parallel to AF .

Feedback

$ED = AF$ given

Angle $BED =$ Angle AFC

$EB = CF$ given

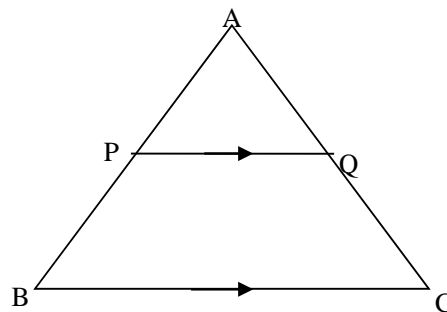
Therefore, $ED \parallel AF$ SAS



Assignment

Exercise 2

1. In this diagram below ABC is a triangle such that PQ is parallel to BC , $BC = 2PQ$ and $BC = 9\text{cm}$.



(a) Show that triangle APQ is similar to $\triangle ABC$

(b) Hence find the length of PQ

Feedback

(a) Lets consider the ratios of BC to PQ, this is shown as $BC = 2 PQ$

This means that BC is twice PQ, which gives as the ratio as $2x : x$.

When written as a ratio this will be 2 to 1 or $2:1$.

(b) since we are given the length of BC to be 9cm then from the ratio

we know that BC is twice PQ then PQ will be $\frac{9}{2}$ hence the length PQ will be 4.5cm

Exercise 3



Assignment

1. A cylinder of radius 8cm has a volume of 128cm^3 . Find the volume of a similar cylinder of radius 6cm.
2. A solid has a height of 15cm and a volume of 360cm^3 . A similar solid has volume 9720cm^3 . Find its height.

Feedback

1. Radius (cm) volume(cm^3)

8 ----- 128

6 ----- x

$$\frac{6}{8} \times 128 = 96\text{cm}^3$$

2. Height (cm) volume (cm^3)

15 ----- 360

x ----- 9720

$$\frac{9720}{360} \times 15 = 305\text{cm}^3$$



Assessment

Tutor Marked Assignment 3 (Unit 7, 8 and 9)

Instructions

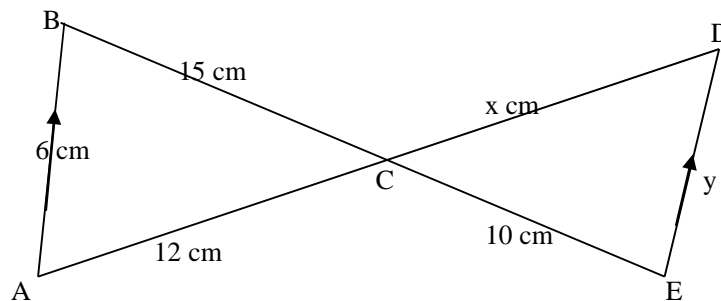
You are required to **Answer all** the questions in the assignment. This assignment is very important to you because it accords you an opportunity to have an overall assessment of the whole study. It also enables you to prepare yourself for your final examinations that you are expected to write at the end of Grade 12.

There five questions in this assignment and carries 25 marks. The marks are shown at the end of each question.

1. Name a polygon that has a 36° rotational symmetry?

Marks: 1

2. In the diagram, AB is parallel to ED. Prove that the triangle ABC and CDE are similar. Hence find the value of x and y.



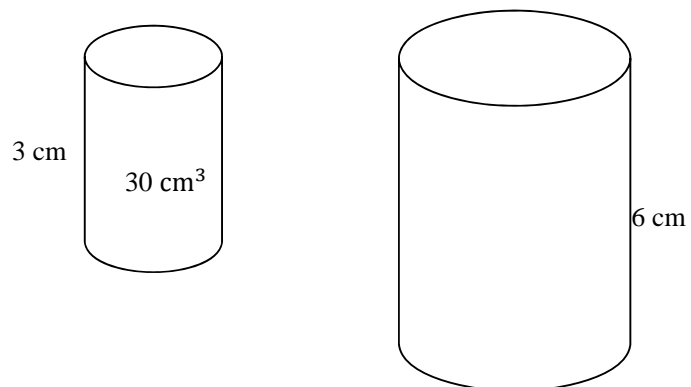
Marks: 4

3. For each of the following shapes, find.

Shape	The number of lines of symmetry.	The order of symmetry.
Square		
Rectangle		
Equilateral triangle		
Kite		
Rhombus		

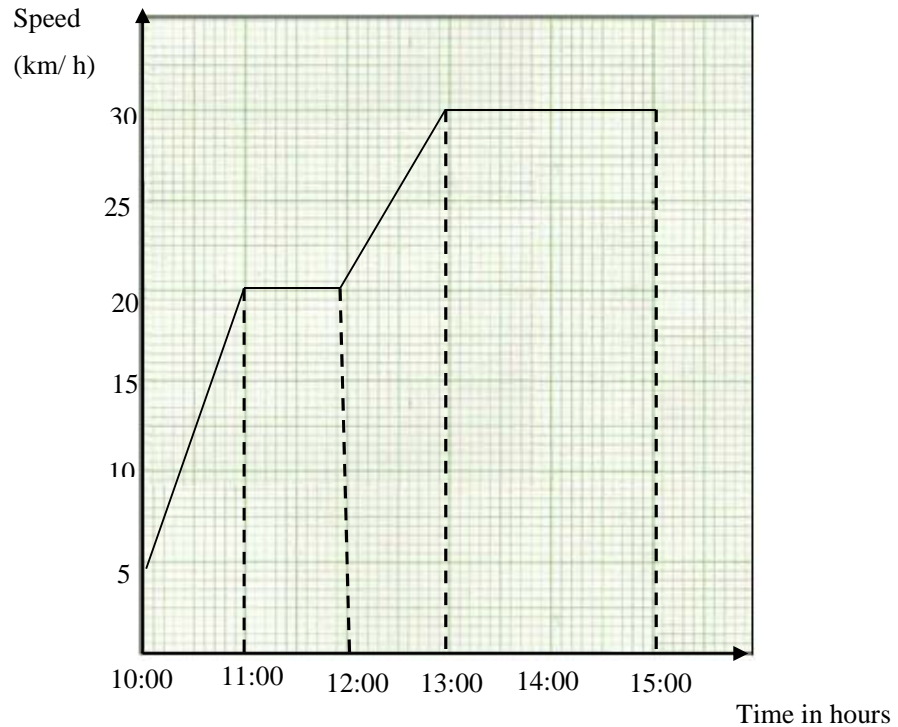
Marks: 10

4. Two similar cylinders have heights of 3 cm and 6 cm respectively. If the volume of the smaller cylinder is 30cm^3 , find the volume of the larger cylinder.



Marks: 3

5. A man went cycling. At 10:00 hours he passed town A, after which he uniformly increased his speed until he reached town B. From town B he cycled at a constant speed until he reached town C. From town C he again increased until he reached town D. From town D he cycled at a constant speed until he reached town E at 15:00 hours. The graph shows his journey



Use the graph to find;

- (i) His speed in km/h as he passed town A
- (ii) The change in speed in km/h as he travelled from A to B
- (iii) The distance between town A and B in kilometres
- (iv) How long he travelled at constant speed altogether
- (v) The change in speed as he travelled from C to D
- (vi) The distance between towns A and E
- (vii) The average speed for the whole journey.

Marks: 7

Marking Key to Tutor Marked Assignment 3

1. $\frac{360^\circ}{36^\circ} = 10$ sides. (a regular 10 sided polygon, **Decagon**).

2. You present corresponding sides as ratios as shown below;

$\frac{15}{10} = \frac{12}{x} = \frac{6}{y}$, since all corresponding sides of similar figures have equal ratios we equate them.

$$\frac{15}{10} = \frac{12}{x}$$

We cross multiply the first pair.

$$15x = 12 \times 10$$

$$\frac{15x}{15} = \frac{12 \times 10}{15}$$

Divide both sides by 15

$$x = 8$$

Similarly, to find for y we follow the method as in finding the value for x.

$$\frac{12}{8} = \frac{6}{y}$$

$$12y = 8 \times 6$$

$$\frac{12y}{12} = \frac{8 \times 6}{12}$$

Divide both sides by 12

$$y = 4$$

Therefore, $\frac{15}{10} = \frac{12}{8} = \frac{6}{4}$, when these are reduced their will give us the same ratio as **3:2**, hence, proof.

3.

Shape	The number of lines of symmetry.	The order of symmetry.
Square	4	4
Rectangle	2	2
Equilateral triangle	3	3
Kite	2	1
Rhombus	0	4

4. $\frac{3}{6} = \frac{30\text{cm}^3}{x}$ Cross multiply the ratios

$$3x = 30 \times 6$$

$$\frac{3x}{3} = \frac{30 \times 6}{3}$$
Divide both sides by 3

Therefore, $3x = \frac{30 \times 6}{3}$

$$x = 60 \text{ cm}^3$$

5. (i) $\text{speed} = \frac{\text{distance}}{\text{time}}$

$$= \frac{5 \text{ km}}{1 \text{ h}}$$

$$= 5 \text{ km/h.}$$

(ii) change in speed = $20\text{km/h} - 5\text{km/h}$

$$= 15 \text{ km/h}$$

(iii) distance town A and town B is the area under the

Graph

$$\text{Area} = 5 \times 1$$

$$= 5 \text{ km}$$

$$\text{Area} = \frac{1}{2}bh$$

$$= \frac{1}{2} \times 1 \times 15$$

$$= \mathbf{7.5 \text{ km}}$$

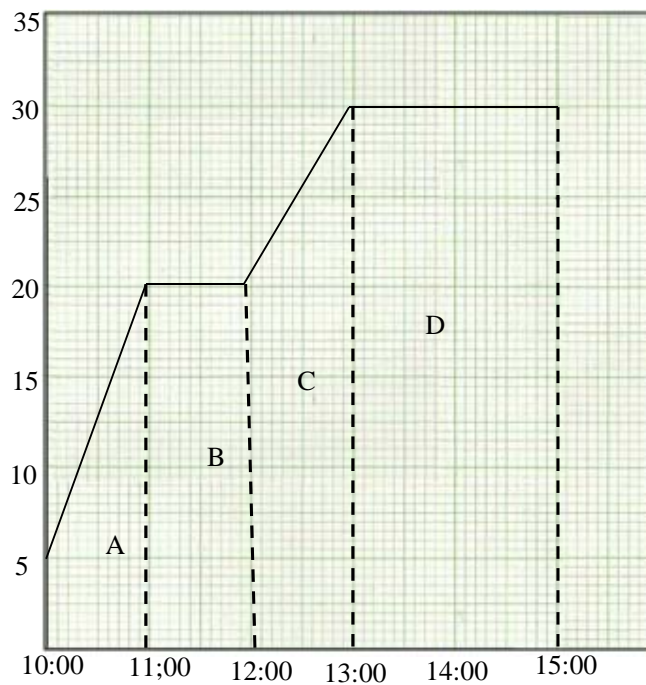
$$\text{Therefore, distance} = 5 \text{ km} + 7.5 \text{ km}$$

$$= \mathbf{12.5 \text{ km}}$$

$$\text{(iv) } 1 + 2 = \mathbf{3 \text{ hours}}$$

$$\text{(v) } 30 \text{ km/h} - 20 \text{ km/h} = \mathbf{10 \text{ km/h}}$$

(vi) the distance of the whole journey is the area under the graph.



$$\text{Area A} = \frac{1}{2} (a + b) h$$

$$\text{Area A} = \frac{1}{2} (5 + 20) 1$$

$$= \frac{1}{2} (25) 1$$

$$= 12.5 \text{ km}^2$$

$$\text{Area B} = l \times b$$

$$= 20 \times 1$$

$$= 20 \text{ km}^2$$

$$\text{Area C} = \frac{1}{2} (a + b) h$$

$$\text{Area C} = \frac{1}{2} (20 + 30) 1$$

$$= \frac{1}{2} (50) 1$$

$$= 25 \text{ km}^2$$

$$\text{Area D} = l \times b$$

$$= 2 \times 30$$

$$= 60 \text{ km}^2$$

$$\text{Total distance} = 12.5 + 20 + 25 + 60$$

$$= \mathbf{117.5 \text{ km}}$$

(vii) Therefore, average speed for the whole journey is;

$$\begin{aligned} \text{Speed} &= \frac{\text{distance}}{\text{time}} \\ &= \frac{60+25+20+12.5}{5} \\ &= \frac{117.5}{5} \\ &= \mathbf{23.5 \text{ km/h}} \end{aligned}$$

Total marks: 25

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Unit 10

Angle Properties of a Circle and Polygons

Introduction

Welcome to Unit10, the Angle properties of a circle and Polygons. In your previous topic, you learned about similarity and congruency of shapes and solids. This topic is not different from the previous one because it is also about the properties of shapes, circles and polygons, but with more emphasis on angles. In this topic, we shall concentrate on the angle properties of the circle and polygons. In your junior course, you covered polygons such as triangles, rectangles, squares and other shapes in the topic Mensuration. You also covered the circumference and area of a circle. The use of angles was discussed when calculating areas of parts of the circle. This means that you have some basic skills about angles, polygons and circles which are the main components of this unit.

The angle properties of a circle and polygons are very important in every day works such as in the construction sector. You may realise that all physical objects are of some shape. The nature of shapes and solids depend on the angles of the vertices and arcs in the objects. This study will make you understand why objects look different from one another.

Just like any other topic in this course, Angle properties of a circle and polygons, is examinable at the end of the programme for your school certificate. You are advised to work through the unit by doing all the activities and exercises given. The unit has two topics namely; Topic 1: Angle properties of a circle and Topic 2: Angle properties of polygons. Topic 1 has five activities and one topic exercise while Topic 2 has six activities and one end of topic exercise. You will not write a Tutor Marked Assignment at the end of this unit. However, you are expected to write the fourth tutor marked assignment at the end of unit 12.

Upon completion of this unit you will be able to:

- *Use* the angle properties of a circle
- *Use* the angle properties of polygons
- Solve mathematical problems involving angles in polygons.



Outcomes

Time Frame



How long?

We estimate that to complete this unit you will need between 10 and 12 hours. This time includes the time you will spend in doing the activities checking them against the feedback. If you do not finish studying the unit within this estimated time do not worry since we do not learn at the same pace.

You are encouraged to spend about 1 hour answering the first topic exercise and 2 hours on the second topic exercise in this unit. Since there are two topic exercises in this unit you are expected to spend 3 hours on them.

The total hours for completing the unit will thus be between 13 and 15 hours.

Learning Resources



Learning Resources

In order to study the Angle properties of a circle and polygons with minimal difficulties you will need the following materials:

- A ruler
- A pencil or a pen
- Protractor
- A pair of compass
- Eraser

Learning and Teaching Approaches

In this unit we have used three teaching and learning methods in presenting the content. These methods are:

- **Conceptual:** This method will help you understand the meaning of facts, rules, formulas and procedures used in Angle properties of a circle and polygons;
- **Problem-solving:** This method will help you solve mathematical problems that relate to real life situations by the use procedures and concepts you shall develop as you go through the Unit. You will also be able to discuss mathematical problems, answers and

strategies with friends.

- **Skills:** Angle properties of a circle and polygons are presented with specific skills on how to problems. These skills will help how to use the facts, rules, formulas as you do self-marked activities and topic exercises.

Terminology

The diagram below the terminology illustrates some of the terms.



Terminology

Arc:	This is the distance around the circle.
Chord:	A line joining two parts on the circumference (a diameter is a special chord).
Circle:	Shape of locus of points of a fixed distant from a fixed point.
Circumference:	This is the distance around the circle.
Diameter:	is a line which passes through the centre This and joins two opposite points on the circumference e.g. A B in the diagram
Exterior angle:	Angle formed outside a polygon between a side and an extended side.
Interior angle:	Angles inside a plain figure or a polygon.
Polygon:	A plain figure bounded by straight lines.
Quadrilateral:	A four sided polygon.
Vertex:	The point where two straight lines meet.
Sector:	This is the area formed by two radii and an arc.
Segment:	The area formed by an arc and a chord (see the diagram).

Some of the above mentioned terms are illustrated in figure 1 below.

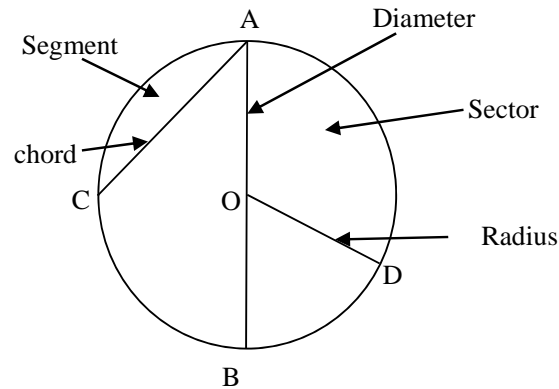


Figure 1: Parts of a circle

Topic 1: Angle Properties of a Circle

Introduction:

Welcome to Topic 1 on Angle properties of a circle. In this topic, we are expected to cover the first outcome of this unit which is to be able to state and *use the angle properties of a circle*.

In our conversation on the angle properties of a circle, we will discuss how these angles are related to each other in a circle. These relationships bring out the properties. For instance, the angle formed at the centre of the circle is related to the angle formed by the same chord at the circumference of the circle. These are the type of relationships which we will discuss. You will learn further about angles in the semi-circle, angles in the same segment and angles in the opposite segment.

In this topic, we are not discussing angles and what they are. We are discussing their properties as they exist in certain situations. This means that your knowledge about angles from earlier works will benefit you very much.

To achieve the first outcome, we have to fulfil the following objectives: in this topic

Objectives



Objectives

- Find angles at the centre and on the circumference of the circle.
- Find angles in a semicircle.
- Find angles in the same segment.
- Find angles in opposite (alternate) segment.
- Find angles in the circle by use of tangents and radii.

You are expected to carry out five activities and one exercise for you to achieve the outcome and objectives mentioned above.

We will now start with the first objective to develop the relationship between the angle at the centre and those in the remaining part of the circumference.

The angle at the centre and the angles on the circumference.

The rule states that the angle made at the centre of a circle is always double or twice the angle at the circumference when the points stand on the same arc. Here, the arc of a circle produces an angle at the centre of the circle and another angle on the remaining part of the circumference. The above can be fully explained by the use of Figure 2 below.

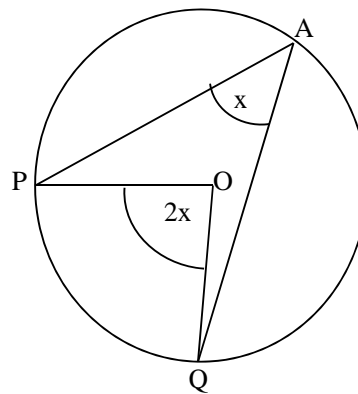


Figure 2: Showing the two angles formed by the same arc, PQ

The circle APQ has an arc PQ which produces an angle POQ at the centre O and angle PAQ on the circumference.

$$\begin{aligned} \text{From the above rule we have; } \widehat{POQ} &= 2 \times \widehat{PAQ} \\ &= 2x. \end{aligned}$$

Let us show the theory above by a mathematical generalisation. For any two angles, one at the centre and another on the circumference of the circle, the generalisation is that the one at the centre is twice the one on the circumference.

We use Figure 3 below for this discussion.

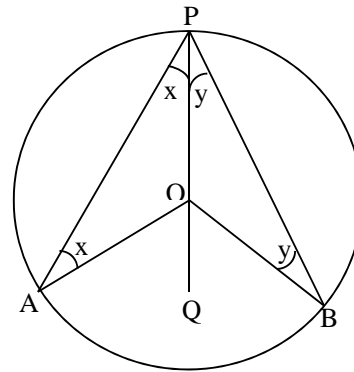


Figure 3: Showing the angle at the centre and on the circumference.

We shall use the knowledge of sum of angles inside a triangle of 180° and the sum of angles at a point of 360° and the base angles of an isosceles triangles.

From Figure 3 above we have,

$$AO = OP = OB \text{ (Radii of same circle).}$$

$$\widehat{APO} = \widehat{OAP} = x \text{ (the base angle of isosceles triangle OAP)}$$

$$\widehat{OPB} = \widehat{OBP} = y \text{ (base angles of isosceles triangle OPB).}$$

We are to show that $\widehat{APB} = x + y$

$$\text{i. } \widehat{AOP} = 180^\circ - 2x$$

$$\text{ii. } \widehat{POB} = 180^\circ - 2y$$

$$\begin{aligned} \text{iii. } \widehat{AOB} &= 360^\circ - (\widehat{AOP} + \widehat{POB}) \\ &= 360^\circ - [(180^\circ - 2x) + (180^\circ - 2y)] \\ &= 360^\circ - (180^\circ + 180^\circ - 2x - 2y) \\ &= 360^\circ - (360^\circ - 2x - 2y) \\ &= 360^\circ - 360^\circ + 2x + 2y \\ &= 2x + 2y \end{aligned}$$

$$= 2(x + y) = 2 \times \widehat{APB}$$

We have shown that the angle at the centre is twice the angle on the circumference as long as they (the two angles) are produced by the same arc.

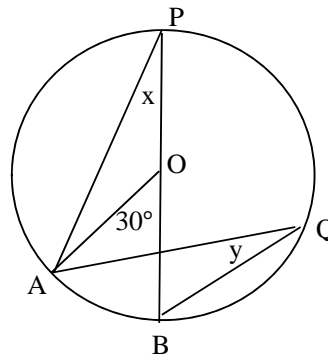
You must apply the property above to work out the activity below. This activity is designed to assist you to learn to use the property in solving problems involving angles. By using the property you will be practicing to use it in situations.



Activity

Activity 1: How to find the angle at the centre and on the circumference which are produced by the same arc.

The figure below shows a circle APQB with centre O. Angle AOB is 30° . Find the angles marked x and y.



Feedback

How did you do you manage? We hope you did well because you could refer to our earlier discussion. Read our feedback and compare it with your answer.

The angle at the centre is 30° and the angles on the circumference are x and y.

Using the property above, this means that $30^\circ = 2 \times x$ and $30^\circ = 2 \times y$.

$$2x = 30^\circ \text{ and } 2y = 30^\circ$$

$$\frac{2x}{2} = \frac{30^\circ}{2} \text{ and } \frac{2y}{2} = 30^\circ$$

$$x = 15^\circ \text{ and } y = 15^\circ.$$

You should notice that the process of working this one is the same as in Figure 3. The angles that are at the centres are twice the angle at the circumference. This means that the angles at the circumference are half the angle at the centre, hence $x = 15$ and $y = 15$.

Let us now move to the next property of the angles in a circle. This property is helpful in understanding the angles formed in a semi-circle and how they can be used in answering questions involving such angles.

The angles in the semicircle

We shall discuss this component by the use of the diagram in Figure 4. This is addressing Objective 2 of Topic 1. This is related to other angles in the circle in the sense that they are formed in the same circle.

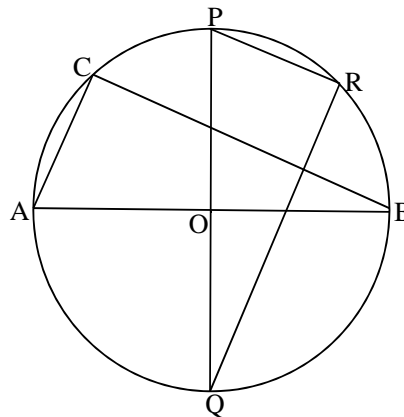


Figure 4: Angles in semicircle.

In Figure 4 above, AOB and POQ are diameters of the circle ACPRBQ with centre O.

A diameter is a line that divides a circle in half. Each half is called semicircle. Let us consider the semicircle ACPRB in which AOB is the diameter.

Let us follow the step;

- i. $\widehat{AOB} = 180^\circ$ (straight line)
- ii. Arc AQB has produced \widehat{AOB} at the centre and \widehat{ACB} on the circumference.
- iii. From the first theory, then $2\widehat{ACB} = \widehat{AOB}$

iv. Then $\widehat{ACB} = \frac{\widehat{AOB}}{2} = \frac{180^\circ}{2} = 90^\circ$.

Similarly

i. $2\widehat{PRQ} = \widehat{POQ}$

ii. $\widehat{PRQ} = \frac{\widehat{POQ}}{2} = \frac{180^\circ}{2} = 90^\circ$.

From the above illustrations we can state that;

- Every angle in a semicircle is a right angle.

Let us consider the next property of the angles in the same segment produced by the same arc. This is the third objective in Topic 1. You should note that all these objectives are related as they define the angle properties of a circle.

You are advised to do this activity in order to establish the property of angles in the same segment. Find a clean sheet of paper on which you should work out this activity.



Activity

Activity 2: Angles in the same segment

Draw a circle with a chord AB.

A chord can be defined as a line that cuts a circle into two parts. Each part is called a segment. It need not pass through the centre.

Mark and label the point P, Q, and R on the circumference above your chord.

Draw straight lines from A to P, Q and R.

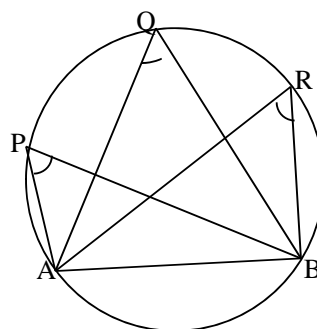
Draw straight lines from B to P, Q and R.

Use a protractor to measure the following angles and write their values down.

- i. \widehat{APB}
- ii. \widehat{AQB}
- iii. \widehat{ARB}

Feedback

Your diagram should have looked something similar to the diagram below.



You must have found the same values for the above angles. This is the case for all circles despite the size of the circles involved. Thus we state:

- Angles subtended (produced) by the same arc or chord in the same segment are equal.

Now work out the following activity to see if you can apply the two theories of angles in the same circle and angles in the same segment.

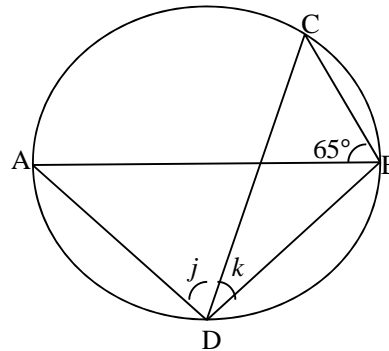


Activity

Activity3: Angles in semicircles and in the same segment.

Study the diagram below and work out the values of j and k . Given that AB is a diameter.

Write the answers in the space below the diagram.



Feedback

We are told that AB is a diameter. So the angle $(j+k)$ is an angle in a semicircle and therefore, it is 90° .

But angle j and 65° are subtended (produced) by the same arc so $j = 65^\circ$.

Then,

$$j + k = 90^\circ$$

$$65^\circ + k = 90^\circ$$

$$k = 90^\circ - 65^\circ$$

$$k = 25^\circ$$

You should note that by using the angles formed in the semicircle, you are able to find the missing or unknown angle. You calculated first the angle j and then angle k .

Let us consider the next property of the angles in opposite segment. Remember that this property is also one of the objectives in this topic, that is, objective number 4. The essence of working this one again is to establish the property. You need to identify the property for you to use them effectively in any situation involving angles in a circle.

Angles in opposite segment

Let us consider the figure below for the concept of opposite segment.

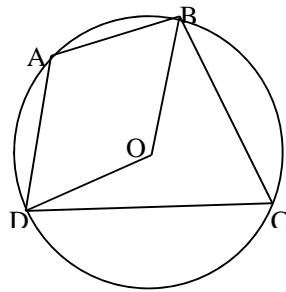


Figure 4(a): Opposite segments.

In Figure 4(a), we have segments BAD and BCD which are opposite to each other.

If we let angle BAD to be x , then reflex angle BOD will be $2x$ (angle at the centre) and if we let angle BCD be y , then similarly angle BOD will be $2y$.

Let us continue by considering angles as depicted in Figure 4(b).

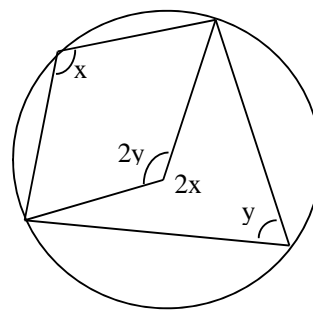


Figure 4(b): Angles in opposite segments.

We now consider the sum of angles at the centre which add up to 360° .

This gives: $2x + 2y = 360^\circ$

$$2(x + y) = 360^\circ$$

$$x + y = 180^\circ.$$

This forms the property that angles in opposite segment are supplementary.

So if we have all the four interior angles for the quadrilateral as shown in Figure 4(c), we may make the following conclusions;

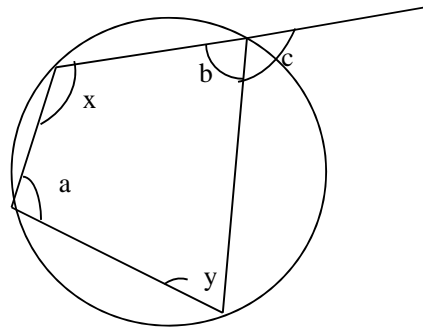


Figure 4 (c): Showing a cyclic quadrilateral

- (i) $x + y = 180^\circ$
- (ii) $a + b = 180^\circ$
- (iii) $c + b = 180^\circ$ (straight line)
- (iv) $a = c$ (*Interior opposite angle equal to exterior angle*)

Let us revise a bit about shapes or figures. We learnt that a polygon is a figure bounded by straight lines. When we learn about polygons, we learn about the different shapes or figures that exist in the world. Among these shapes, we have the triangle, the rectangle and the square.

Now, we also learn that each of these figures falls in a category that is formed. Each of the categories is formed by considering the number of sides a figure has. A figure with three sides is put in a category called triangles. Any figure with four sides is put in the category called quadrilaterals.

A quadrilateral inside a circle is called a cyclic quadrilateral. The following observations form the properties in general.

- Opposite angles of cyclic quadrilateral are supplementary.

- An exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

Now do the following activity to assess how well you have understood the properties of angles in opposite segment of a cyclic quadrilateral.

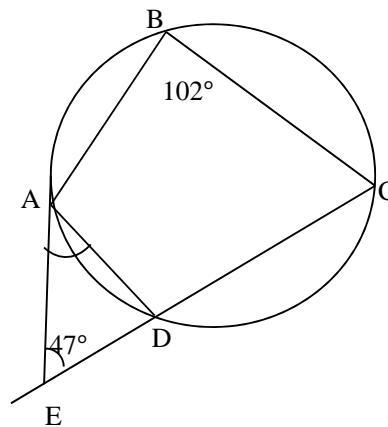


Activity

Activity 4: Angles in opposite segment.

In the diagram below, A, B, C, D are points on a circle such that $\widehat{ABC} = 102^\circ$.

CD is produced to so that $\widehat{AED} = 47^\circ$. Calculate \widehat{EAD} .



Feedback

You should have considered the properties of a cyclic quadrilateral. For the first angle, you should have considered supplementary property as shown below:

102° and \widehat{ADC} (Opposite angles of a cyclic quadrilateral)

These angles are supplementary, which means that they add up to 180°

$$102^\circ + \widehat{ADC} = 180^\circ$$

$$\widehat{ADC} = 180^\circ - 102^\circ$$

$$\widehat{ADC} = 78^\circ$$

Note: $\widehat{ADE} = \widehat{ABC}$

$$\widehat{ADE} = 102^\circ$$

$$\widehat{EAD} = 180^\circ - (102^\circ + 47^\circ) \text{ [Angle sum in triangle]}$$

$$\widehat{EAD} = 180^\circ - 149^\circ$$

$$E\hat{A}D = 31^\circ$$

You should note that we have calculated the value of the angle by using the property. This should tell you that understanding the properties will enable you to work with these types of angles in any situation.

Let us now move to the next property of angles in alternate segment. This one is the last of the objectives in this topic. You should note that these properties work together under the name angle properties in a circle.

Angles in Alternate segment

We first define two principle lines, the chord and the tangent, for us to continue with the concept of angles in alternate segment.

A chord was defined earlier on. However, we shall redefine it for emphasis sake. A chord is a straight line which cuts a circle into two segments. A tangent to a circle can also be defined as a straight line which meets or touches a circle at one point.

We now look at the concepts of alternate segment and angles in alternate segment. Two segments are said to be alternate to each other when a tangent meets the chord at its point of contact.

The figure below shows angles in alternate segment.

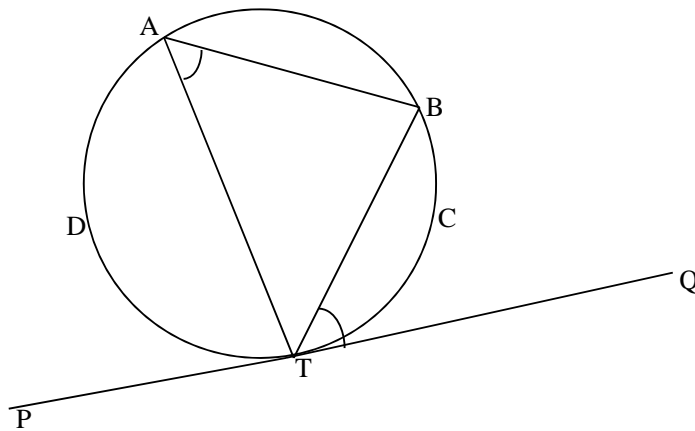


Figure 5: Alternate Segment

In Figure 5, the segments TBC and TDAB are alternate segments and angle TAB and angle BTQ are angles in alternate segment. In these situations, you will see that the alternate segments are formed by chords. The chords in this diagram are AT, AB and BT. For each chord, we have alternate segments. Now, the line PTQ is a tangent to the circle. By tangent to the circle we mean the line that touches a circle at only one point of meeting. To remind yourself about this information, check with the information in the terminology section at the beginning of this topic.

Now do activity below to establish the property of angles in the alternate segment.



Activity

Activity 5: Angles in Alternate segment.

- (i) Draw a diagram similar to the figure below to investigate the relationship between angles in alternate segment.

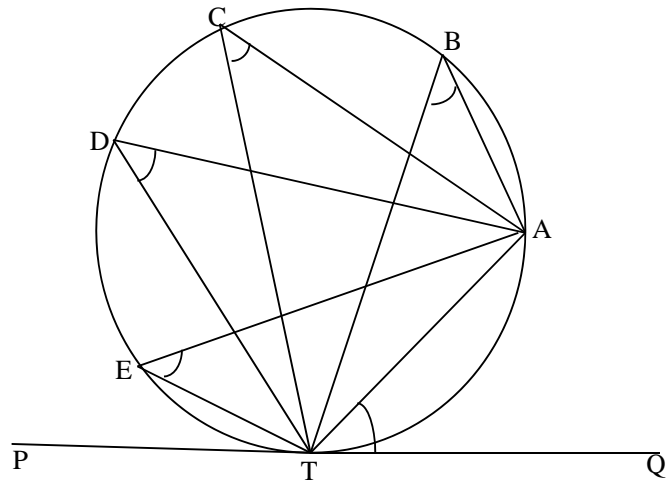


Figure 6: Angles in Alternate segments.

- (ii) Name the angles found in the alternate segment to the segment which has \widehat{ATQ} .

- (a) _____
- (b) _____
- (c) _____
- (d) _____

- (iii) Measure and write down the value of \widehat{ATQ} .

\widehat{ATQ} _____.

- (iv) Write down the values of angles in (ii)

- (a) _____
- (b) _____
- (c) _____
- (d) _____

- (v) Compare the value of angle \widehat{ATQ} to the values you have found in (iv) and make a comment.

Feedback

You can now compare your answers with ours. You should have:

- (i) *The Diagram as presented in the instructions.*
- (ii) *First you should have identified the chord that makes the two segments. This chord is AT . The one segment is below AT and the other is above AT . These two are alternate segments to each other. You should have written down the angles in the segment above AT . These angles are: $\widehat{TBA}, \widehat{TCA}, \widehat{TDA}, \widehat{TEA}$. \widehat{ATQ} is found in the segment below AT . We say that \widehat{ATQ} is found in the alternate segment to the segment containing $\widehat{TBA}, \widehat{TCA}, \widehat{TDA}, \widehat{TEA}$.*
- (iii) *You should have measured the angle \widehat{ATQ} . Since the diagram comes out depending on the person drawing it, the value of angle \widehat{ATQ} will depend on your diagram.*
- (iv) *The above situation applies to this part of the question. You should have measured the angles. The values will be according to your diagram but they are all equal.*
- (v) *Comparing the measured angles in (iv), we observe that they are equal to each other. This is derived from Property Number 3. You should also have observed that these angle values in (iv) are all equal to the value of \widehat{ATQ} . Therefore, we say that angles in alternate segments are equal.*

The theorem is hence stated: Angles in alternate segments are equal.

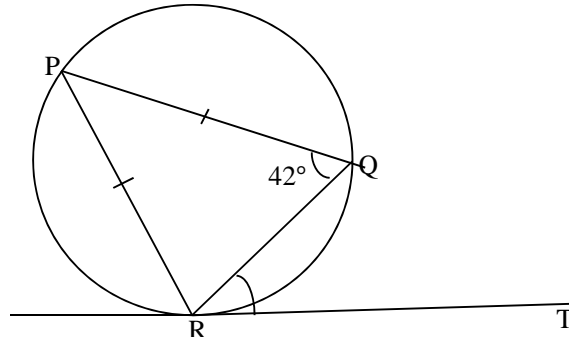
Now work out the Activity 5 so that you make use of the property stated above. The above activity was for you to work out a general situation with no specific numbers. This helps you to appreciate the property better as you can use it in any situation. The following activity is designed to help you use figures or specific values in using the property. It may seem that you are repeating the activity but you are now doing it with actual values of angles.



Activity

Activity 6: Angles in Alternate segment.

In the diagram below RT is a tangent to the circle RPQ. If $\widehat{RQP} = 42^\circ$, and $RP=PQ$. Calculate \widehat{QRT} .

**Feedback**

After the previous activity and the discussion in the feedback, you should have found this activity easier to do. If not, after you have read this feedback, you may have to revise this section.

Since triangle RPQ is isosceles then the base angles are equal. Thus $\widehat{QRP} = 42^\circ$.

$$\text{Then } \widehat{RPQ} = 180^\circ - (\widehat{PQR} + \widehat{QRP}) = 180^\circ - (42^\circ + 42^\circ)$$

$$\widehat{RPQ} = 180^\circ - 84^\circ = 96^\circ.$$

Since, $\widehat{RPQ} = \widehat{QRT}$. (Angles in alternate segment).

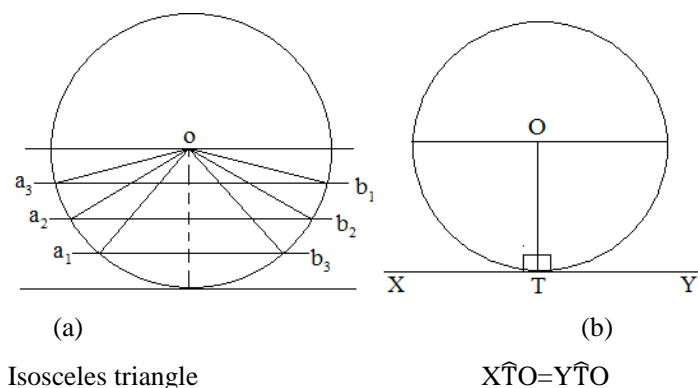
$$\widehat{QRT} = 96^\circ$$

We now move to the last property which is on Tangent properties to the circle. You should have noted in the last activity that there was a tangent line mentioned. You were asked to find one angle formed by the tangent and the chord, which was angle QRT . We have to use the tangent property to find the angle. This property will assist you be able to find the angle when a tangent is involved. This addresses the last objective.

The Tangent to the Circle

We will state the property without proof (property to be explained by use of diagrams) that a tangent meets the radius at right angle. This property can be explained from the movement of a straight line parallel to the diameter until it becomes a tangent to the circle as shown in the figure below.

Figure 7: Tangent to the circle



Let us consider Figure 7(a) in which the chords a_1b_1 , a_2b_2 , a_3b_3 and so on moves parallel away from the diameter until it becomes the tangent to the circle. The isosceles triangle, oab , has base side, ab , as the chord. The triangle has equal sides (radii) and equal base angles move into different positions. The base angles are equal at every position, that is, angle oab = angle oba . When the chord becomes the tangent, XTY , the equal sides (radii), oa and ob , coincide in one radius, OT , meeting a straight line, the tangent. The two equal base angles form the straight angle (180°).

Since the base angles are equal then each base angle is 90° . Thus, the tangent to the circle meets the radius at right angle as illustrated in Figure 7 (b).

The property states in full that: The tangent to a circle meets the radius of that circle at right angle (90°). It can also be stated that any straight line from a circle that meets the tangent at right angle passes through the centre of the circle.

The following is not a separate property but falls within the property of tangent to a circle. This may seem to be different. In this part we are considering tangent from an external point and their effect on the circle and angles formed in the circle.

The Tangents from an External Point

Let us consider the diagram below for establishing the properties of two tangents from an exterior point.

Figure 8: Two Tangents to the circle from a fixed point.

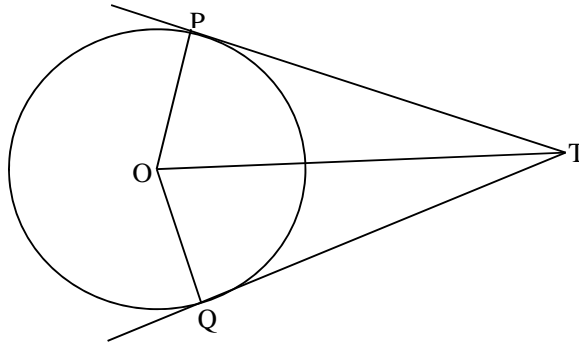


Figure 8 shows two tangents TP and TQ meeting the circle at P and Q. The fixed distance from O the centre of the circle is OT.

Now we use the theory of congruency from your Unit 9 to show properties of two tangents from a common point.

- (i) $\hat{TPO} = \hat{TQO}$ (Radius meets tangent at right angle).
- (ii) $PO = QO$ (radii of same circle).
- (iii) $OT = OT$ (common side to both triangles)

From the property of congruency we use Right angle, Hypotenuse, Side (RHS). Therefore, $\Delta TPO \cong \Delta TQO$. We can conclude from congruency that $PT = QT$ and $\hat{TPO} = \hat{TQO}$.

We now state the property that:

Two tangents from an exterior point to the circle are equal in length up to the point they touch the circle and the angles they make with the line from the centre and the fixed source of tangents are equal.

The figure below shows a circle, centre O, and tangents PT and QT from a fixed point T.

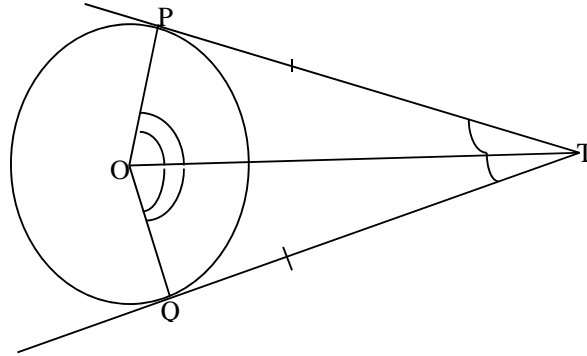


Figure 9: Tangents from a fixed point to a circle.

Using what we have so far learnt from the property of tangent, we can consider the above diagram.

In the diagram, tangent PT is equal to tangent QT. We have also discussed the angles formed by the tangents. So angle OQT is equal to angle OPT. These are angles formed by the two tangents from the external point. It also follows that angle QTO is equal to angle PTO.

You should note that since $\widehat{O\hat{T}Q} = \widehat{P\hat{T}O}$, $TQ = TP$ and that $\widehat{T\hat{Q}O} = \widehat{T\hat{P}O}$, then it follows that the triangles, OTP and OTQ are the same. Such triangles we say that they are congruent. This means also that the line TO forms two congruent triangles; hence it bisects the angles, $\widehat{Q\hat{T}P}$ and $\widehat{Q\hat{O}P}$.

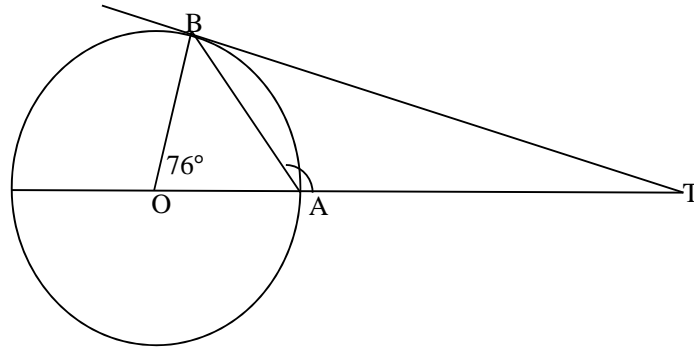
You have learned the properties of tangent to the circle. Work out the activity below to see whether you can apply these properties to solve the problems.



Activity

Activity 7: Tangent Properties to the Circle.

Given that O is the centre of a circle and TB is a tangent at B. Calculate $\hat{A}BT$, if $\hat{B}OA = 76^\circ$. Use the diagram below.

**Feedback**

We start by using the isosceles triangle OBA. You should have found that:

$$\hat{O}AB = \hat{O}BA$$

These are base angles of an isosceles triangle. Hence, they are equal. Since we know that the sum of angles in a triangle is 180° , you should have added the three angles of the triangle OBA as follows:

$$2\hat{O}BA + 76^\circ = 180^\circ$$

You should have worked to make $\hat{O}BA$ the subject of the formula in the statement as follows:

$$2\hat{O}BA = 180^\circ - 76^\circ$$

$$\hat{O}BA = \frac{104^\circ}{2} = 52^\circ$$

This shows that the angle OBA = 52° .

We also know that the angle formed by the tangent to the circle and the radius is 90° .

$$\text{Therefore, } \hat{O}BT = 90^\circ$$

Using the triangle OBT, you should have calculated for $\hat{A}BT$ as follows:

$$\hat{A}BT + \hat{O}BA = 90^\circ$$

$$\hat{A}BT = 90^\circ - \hat{O}BA$$

$$\hat{A}BT = 90^\circ - 52^\circ = 38^\circ.$$

$$\hat{A}BT = 38^\circ.$$

Topic1 Summary



Summary

In this Topic1 you learned the angle properties of the circle .You have covered the first outcome. This outcome was broken down into five objectives. Each of these objectives addressed a particular property. These properties included the Angle at the centre in which you learnt that an arc subtends an angle at the centre which is twice the angle it subtends on the remaining part of the circumference. The second stated that the angle produced by the semicircle is a right angle You also learned that angles in the same segment produced by the arc are equal. You covered that a chord divides a circle into two segments which are alternate to each other and that angles in the alternate segment are equal. Lastly you looked at the properties of tangents to the circle that a tangent meets the radius at right angle and that two tangents from a common point have equal length. You also discovered that a line through the centre of the circle and the source of the tangents form two congruent triangle and bisect the angle between the tangents.

You have come to the end of Topic 1. Answer Topic Exercise 1 in the assignment section. After answering the exercise and checking your answers, you should proceed to the next topic if your progress is satisfactory. If you found some sections difficult, revise these and discuss with other learners and your teacher to found solutions to your challenges.

Topic 2: Angle Properties of Polygons

Introduction

Welcome to Topic 2 on the angle properties of the polygons. In this topic, we shall cover the second outcome of this unit, which is; to state and “*use the angle properties of polygons to solve problems associated with angles in polygons*”.

You will learn about a polygon and its properties. This basically introduces you to the shapes that you may come across in life. There are different shapes in this world but they fall under different classes or rather classification depending on the number of sides in the shape. You will concentrate your learning on 2 dimensions shapes. These have only 2 dimensions, that is, length and width.

In this topic, you will learn about the angles formed in these shapes and how they are related to each other. These angles help you calculate the missing angles in the shape. You will learn that the number of sides of a polygon is equal to the number of angles found in that polygon. This is important information you need to pay particular attention to as it will help greatly in understanding this topic..

In order to achieve the above outcome, you are expected to carry out a number of activities and one exercise.

By the end of this topic you should be able to address the following objectives:

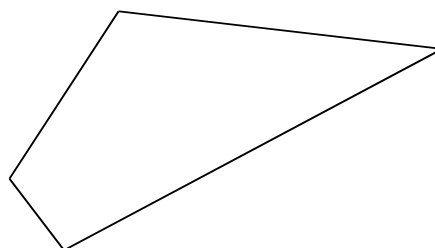


Objectives

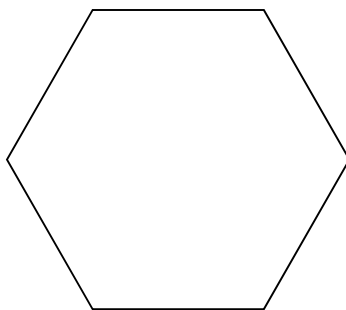
- Find angles using angle properties of triangles and quadrilaterals.
- Find the angle sum of polygons.
- Find the size of interior and exterior angles of a polygon.
- Solve problems involving angle properties of a polygon

We start by looking at the general definition of a polygon. A polygon is a plain figure bounded by straight lines. From your Junior Course in Grade 9, you should be able to recall the names of a number of polygons. Some of the polygons you studied included the triangle (3 sided), the quadrilateral (4 sided), the Pentagon (5 sided) and many others. The polygons are in two categories; **Irregular** and **Regular** polygons. The regular polygons have equal sides and equal angles. The irregular polygons do not have equal sides or equal angles.

Irregular polygons look as shown below:



Regular polygons, on the other hand, look as shown below:



In this topic, we will discuss the properties of angles for the general polygons starting with triangles and quadrilaterals.

But before we go any farther, do the activity below so that you remind yourself on the names of polygons from your Grade 9 work on polygons.



Activity

Activity 1: The polygons

Name the polygon which has the following given number of sides;

- (i) Six ; _____
- (ii) Seven; _____
- (iii) Eight: _____
- (iv) Nine: _____
- (v) Ten: _____

Feedback

You should have named the polygons basing on the number of sides. This is how the naming of the polygons should be:

- (i) *Hexagon* (ii) *Heptagon* (iii) *Octagon* (iv) *Nonagon*
- (vi) *Decagon*

The first name is Hexagon which is a shape with six sides. The next one should be Heptagon which is a seven-sided shape. The next should be Octagon which is an eight –sided shape. The next should be Nonagon from the nine-sided shape. The last one is from ten which is Decagon.

Let us now start with the first objective of angle properties of triangles and quadrilaterals.

Angle Properties of Triangles and Quadrilaterals**The Triangle**

As already stated, a triangle is a three-sided plain shape.

Let us consider the figure below to discuss the angle sum of the interior angles of a triangle.

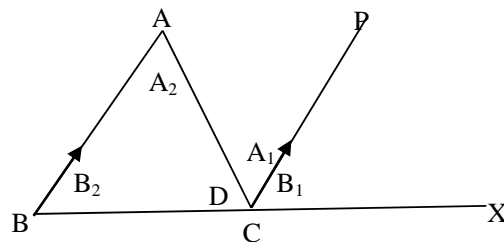


Figure 10: Interior angles of a Triangle.

Here, we have triangle ABC and angles D, A_1 , B_1 along a straight line BCX. Inside the triangle we have angles A_2 , B_2 and D.

The sum of the interior angles of the triangle in figure 10 is given as follows: $A_2 + B_2 + D = ?$.

Now, we know that the sum $A_1 + B_1 + D = 180^\circ$ (angles on a straight line)

Since AB and CP are parallel lines, we have the following:

$$A_1 = A_2 \text{ (alternate angles are equal in parallel lines)}$$

$$B_1 = B_2 \text{ (corresponding angles are equal in parallel lines)}$$

$$D = D \text{ (common angle to straight line and the triangle).}$$

Now, since the sum $A_1 + B_1 + D = 180^\circ$ (angles on a straight line) and these angles (A_1 , B_1 , and D) are respectively equal to the angles (A_2 , B_2 , and D) in the interior of the triangle, then it follows that the sum of angles in the triangle is the same. Therefore, the sum: $A_2 + B_2 + D = 180^\circ$. Thus we state that “the sum of the interior angles of a triangle is 180° ”.

From this, we should now conclude that, the sum of the interior angles of any triangle is 180° .

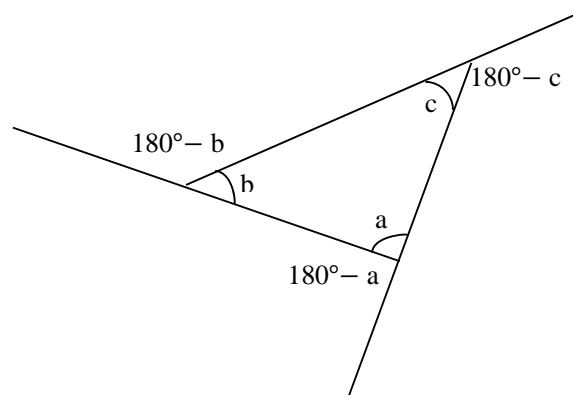
The Exterior Angle

In any polygon, an exterior angle is formed when one side is extended.

The exterior angle and the interior angle are both along a straight line.

From the knowledge of angles on a straight line, their sum is 180° . Let us look at the figure below to demonstrate the stated property.

Figure 11: Exterior angles in polygons.



From figure 11 we may take the following:

The exterior angle to angle $a = 180^\circ - a$

The exterior angle to angle $b = 180^\circ - b$

The exterior angle to angle $c = 180^\circ - c$.

If we add all the exterior angles we have:

$$(180^\circ - a) + (180^\circ - b) + (180^\circ - c) = 540^\circ - (a + b + c)$$

However, we know that the sum of angles in a triangle is 180° . Therefore, $a + b + c = 180^\circ$.

Hence, $540^\circ - (a + b + c) = 540^\circ - 180^\circ = 360^\circ$.

Therefore, $(180^\circ - a) + (180^\circ - b) + (180^\circ - c) = 360^\circ$.

The conclusion is that the sum of the exterior angles of a polygon adds up to 360° .

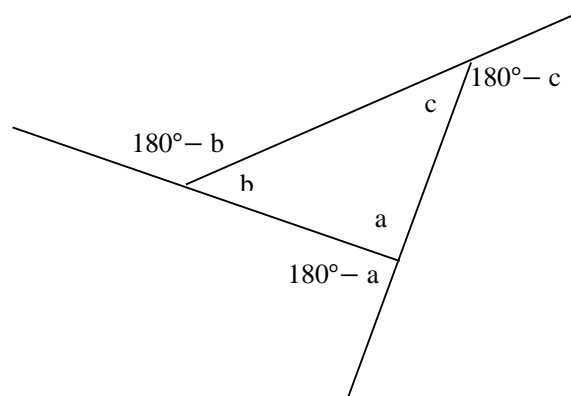
Although we have only used one example, the result is true to all the polygons.

The property states that the sum of exterior angles of any polygon is 360° .

Let us now discuss the last property based on a triangle, that is, the sum of the opposite interior angles to the given exterior angle.

We shall again consider using Figure 11 above in this property.

Let us look at the exterior angle of \hat{a} which is $180^\circ - a$.



We know that the sum of the interior angles: $a + b + c = 180^\circ$.

Let us now take the exterior angle; $180^\circ - a$

The opposite interior angles to this angle ($180^\circ - a$) are b and c .

Sum of the two angles is: $b + c$.

Now from the equation, $a + b + c = 180^\circ$, we find the sum; $b + c$ to be equal to $180^\circ - a$.

We can state this as: $b + c = 180^\circ - a$. This gives us the property which states that;

An exterior angle is equal to the sum of the two opposite interior angles in a triangle.

This means that to find the exterior angle in a triangle, we simply find the sum of the two opposite angle inside the triangle.

Let us now show this information using some different diagram as shown below:

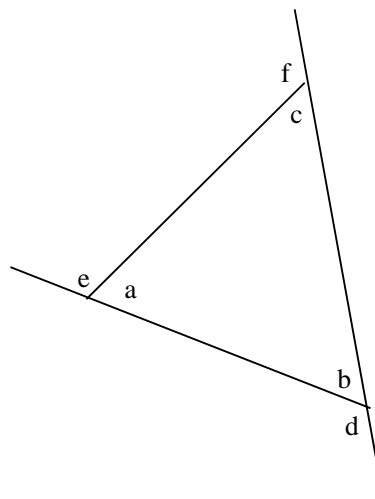


Figure 12: Another example

From this diagram, you should note the following:

$$b + d = 180^\circ$$

$$c + f = 180^\circ$$

$$a + e = 180^\circ$$

However, for the property we have discussed so far, this is what we have:

$$a + b + c = 180^\circ \text{ (sum of angles in a triangle)}$$

When we substitute each of the above three in this equation, we have the following:

$$a + b + c = 180^\circ$$

$$b + d = 180^\circ$$

$$a + b + c = b + d$$

We make d the subject of the formula:

$$a + b + c - b = d$$

$$a + b - b + c = d$$

$$a + c = d$$

$$d = a + c$$

The external angle (d) is equal to the sum of the two opposite interior angles ($a + c$).

Let us now do the next external angle from the diagram.

$$a + b + c = 180^\circ$$

$$c + f = 180^\circ$$

$$a + b + c = c + f$$

We make f the subject of the formula:

$$a + b + c - c = f$$

$$a + b = f$$

$$f = a + b$$

The external angle (f) is equal to the sum of the two opposite interior angles ($a + b$).

Let us also do the last angle external angle from the diagram.

$$a + b + c = 180^\circ$$

$$a + e = 180^\circ$$

$$a + b + c = a + e$$

We make e the subject of the formula:

$$a + b + c - a = e$$

$$a - a + b + c = e$$

$$b + c = e$$

$$e = b + c$$

The external angle (e) is equal to the sum of the two opposite interior angles ($b + c$).

Each of these three above, the external angle is equal to the sum of the opposite two interior angles.

We have discussed angles in a triangle. We have just learnt about external angles and their relationship to the internal angles of the same triangle.

Now, let us look at some triangles and see what makes them different from each other. Here we will basically mention them and state the differences. The angle properties have been discussed above. We need to identify the different triangles. You should note that the angle properties of a triangle apply to all triangles.

The four types of triangle

There are four important types of triangles. These include the following;

(i) **The scalene:**

This triangle has all the sides and angles of different sizes.

The figure below shows a scalene triangle in which all the sides have different lengths and all the angles have different sizes.

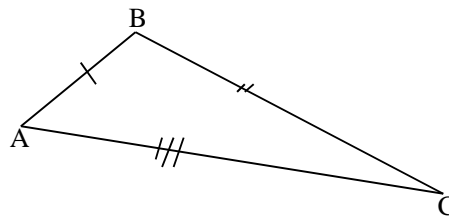


Figure 13: The scalene triangle ABC.

(ii) **The Isosceles triangle:**

The isosceles triangle has a pair of equal angles and a pair of equal sides. The third side is called the base and the equal angles are called the base angles.

Let us use the diagram below to discuss the information of an isosceles triangle.

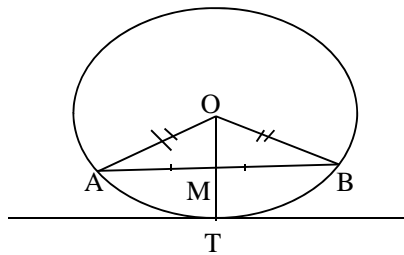


Figure 14: The isosceles Triangle

We use the congruency to explain the properties of isosceles triangle.

We have; $AD=BO$ (radii)

$AM=BM$ (perpendicular bisector to tangent)

$$\widehat{OMA} = \widehat{OMB} \text{ (right angle)}$$

Using the property of Right angle, Side, Side (RSS) of congruency, we can conclude that Triangle OAM is congruent to Triangle OBM.

Since $\triangle OAM \cong \triangle OBM$, we can conclude that $\widehat{OAM} = \widehat{OBM}$.

These angles are called the base angles, OA and OB are equal sides and AB is the base side.

(iii) The Equilateral triangle

This is a special type of isosceles triangle in which all the sides are equal and angles are each equal to 60° . The figure below shows the sides and angles of an equilateral triangle.

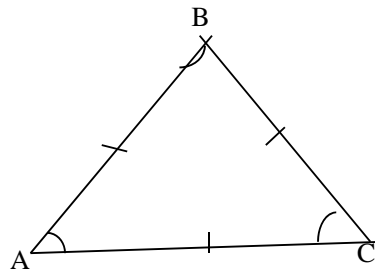


Figure 15: Equilateral Triangle

Figure 15 shows an equilateral triangle in which $AB = BC = AC$ all sides equal in length. And $\widehat{ABC} = \widehat{BAC} = \widehat{ACB} = 60^\circ$.

All angles are equal.

Now do the following activity to assess how you can apply the angle properties on triangles.

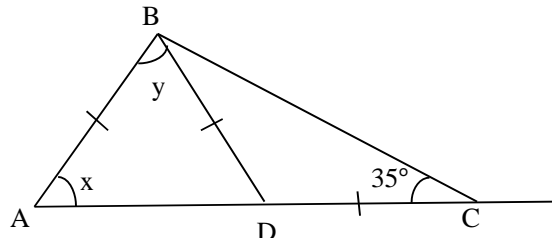


Activity

Activity 2: Isosceles Triangle

Use the diagram below to answer the following questions;

- Name two isosceles triangles
- Find the values of x and y .

**Feedback**

- (a) Using the information we have just learnt on isosceles triangles, you should have noted that the following are the isosceles triangles from the diagram:

Triangle ABD and triangle BDC

In these triangles, we see that $AB = BD$, and $BD = DC$ and that all these sides mentioned are equal.

- (b) Using the same information on an isosceles triangle, $\widehat{C\hat{B}D} = 35^\circ$ (base angles in an isosceles triangle)

Using the property of sum of angles in a triangle:

$$\widehat{D\hat{C}B} + \widehat{C\hat{B}D} + \widehat{B\hat{D}C} = 180^\circ$$

But we know that $\widehat{C\hat{B}D} = \widehat{D\hat{C}B} = 35^\circ$,

$$35^\circ + 35^\circ + \widehat{B\hat{D}C} = 180^\circ$$

$$\widehat{B\hat{D}C} = 180^\circ - (35^\circ + 35^\circ)$$

$$\widehat{B\hat{D}C} = 180^\circ - 70^\circ = 110^\circ$$

$$\widehat{B\hat{D}C} = 110^\circ$$

Using this angle, $\widehat{B\hat{D}C} = 110^\circ$, we can calculate angle BDA.

$$\widehat{A\hat{D}B} = 180^\circ - 110^\circ = 70^\circ$$

$$\widehat{A\hat{D}B} = 70^\circ$$

Now, we also know that triangle ABD is an Isosceles triangle.

Hence, $\widehat{BAD} = \widehat{ADB} = 70^\circ$

Therefore, $x = 70^\circ$.

Now, sum of angles in triangle $ADB = 180^\circ$

$\widehat{BAD} + \widehat{ADB} + \widehat{ABD} = 180^\circ$

$70^\circ + 70^\circ + \widehat{ABD} = 180^\circ$

$140^\circ + \widehat{ABD} = 180^\circ$

$\widehat{ABD} = 180^\circ - 140^\circ$

$\widehat{ABD} = 40^\circ$

Therefore, $y = 40^\circ$

You should note how the information learnt is used in finding values of angles that are missing in questions.

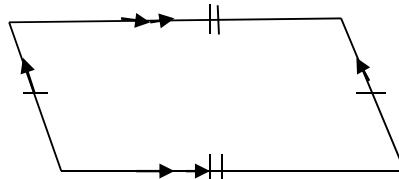
Let us now look at quadrilaterals. You should be reminded that these are also polygons. However, these are four sided figures very different from the triangles which are three sided figures.

The Quadrilaterals

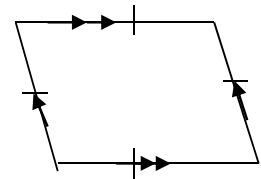
As already stated, a quadrilateral is a four sided polygon. Some common quadrilaterals include the following;

- (i) A parallelogram: This is a quadrilateral with two pairs of sides parallel and equal.
- (ii) A Rhombus: is a parallelogram with all sides equal in length.
- (iii) A Rectangle: is a quadrilateral two pairs of equal sides and all sides meet at right angles.
- (iv) A square: is a rectangle with all its sides equal in length.
- (v) A trapezium: is a quadrilateral with only a pair of parallel sides.

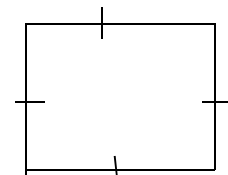
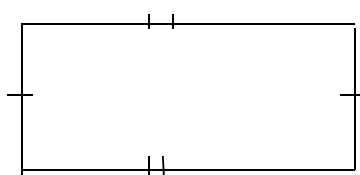
The figures below show the diagrams for the named quadrilaterals.



(a) Parallelogram

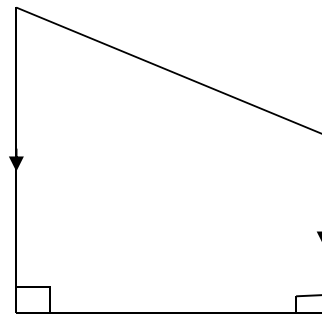


(b) Rhombus



(c) Rectangle

(d) Square



(e) Trapezium

Figure 16: Some examples of quadrilaterals

A square is the only regular quadrilateral since it has all its sides and angles equal in size.

We have just looked at what the quadrilaterals are. We have mentioned them and have seen the different types of quadrilaterals. They have different names but are all four sided figures. Now, we need to see the relationship of angles in the quadrilaterals and how we use these angles in solving problems that are dealing with angles. The next section is dealing with the angles in the quadrilaterals.

The Interior and Exterior Angles of Quadrilaterals

Let us consider one regular polygon, a square, for this discussion. As already stated that all angles are right angles, we extend the sides to produce the exterior angles.

The figure below shows the exterior and interior angles in a square.

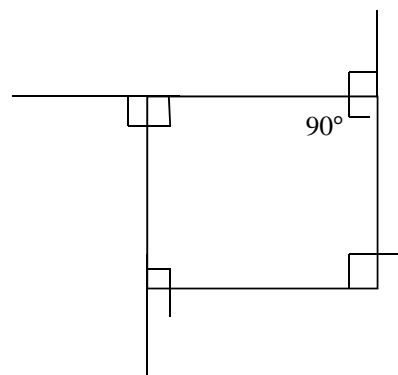


Figure 17: Exterior and Interior angles

In figure 17, we have the interior angle $a = 90^\circ$ and the exterior angle $180^\circ - a$. However, the exterior angle is $180^\circ - a = 180^\circ - 90^\circ = 90^\circ$.

The sum of all exterior angles = $90^\circ \times 4$ (4 exterior angles).
 $= 360^\circ$

The sum of all interior angles = $90^\circ \times 4$ (4 interior angles).
 $= 360^\circ$.

Note: As in the case of triangles, the sum of exterior angles is always 360° . This is a complete revolution by going around a given shape. This applies to all quadrilaterals.

We have now done the angle properties of triangles and quadrilaterals. Let us now move to the general polygons and learn how to find the exterior and interior angles.

The sum of interior angles of polygons

You may have noticed the following in the polygons we have discussed so far that:

- (i) The number of sides in a polygon is the same as the number of angles (both exterior and interior).
- (ii) The sum of exterior angles is always 360°
- (iii) The sum of exterior angle and interior angle is 180° . Both the exterior and interior angles fall on a straight line.

The observations above shall help us to find:

- The exterior angles
- The interior angles
- The sum of the interior angles
- The polygon.

Let us look at an example so that we practise and apply the above properties.

Example 1

A regular polygon has 3 sides. Find:

- (a) Its exterior angle
- (b) Its interior angle
- (c) The sum of its interior angles
- (d) Given that the triangle is right angled and one other angle is 70° . Find the third angle.

Solution

Assuming that the polygon is regular let us apply the principles of regular polygons.

- (a) Since the polygon has 3 sides, it follows that the polygon has 3 exterior angles and 3 interior angles; **check (i) on sum of interior angles of polygon on page 38).**
- (b) To find the size of the exterior angle, we divide 360° by the number of sides of the polygon. Note that the 360° is the sum of the exterior angles of a polygon.

Then, one exterior angle is $\frac{360^\circ}{3} = 120^\circ$.

- (c) The sum of *exterior angle and interior angle* is 180° . (refer to (iii) on page 38.

$$\text{Then, } \textit{exterior angle} + \textit{interior angle} = 180^\circ.$$

$$120^\circ + \textit{interior angle} = 180^\circ.$$

$$\textit{interior angle} = 180^\circ - 120^\circ = 60^\circ.$$

- (d) Since the polygon has 3 interior angles, then

$$\text{The sum of interior angles} = 3 \times 60^\circ$$

$$= 180^\circ$$

- (e) In this question, the triangle is a scalene triangle as it has all three angles different in size. One angle is 90° and the other is 70° . This shows that the triangle is not a regular polygon though the sum of interior angles is still 180° . We let the third angle be x .

$$\text{Thus, } 90^\circ + 70^\circ + x = 180^\circ$$

$$160^\circ + x = 180^\circ$$

$$x = 180^\circ - 160^\circ$$

$$x = 20^\circ$$

The third angle is 20°

From the example above, we can conclude as follows:

For a regular polygon of n sides with an exterior angle, the following stand:

- Exterior angle = $\frac{360^\circ}{n}$
- Exterior angle + Interior angle = 180°

$$\text{Interior angle} = 180^\circ - \textit{exterior angle}$$

- Sum of Interior angles = $n \times \text{Interior angle}$

You should now do the activity below to practice the given principles and formulae.



Activity

Activity3: Exterior and Interior angles of regular polygon

A regular polygon has five sides.

Find (a) one of its exterior angles

- One of interior angles
- The sum of its exterior angles.
- If another polygon has 5 sides and the sum of its 4 angles is 500° .

Find the remaining angle.

Feedback.

$$\begin{aligned} \text{(a) Since the polygon has 5 sides, then Exterior angle} &= \frac{360^\circ}{5} \\ &= 72^\circ \end{aligned}$$

$$\text{(b) The Interior angle} + \text{Exterior angle} = 180^\circ$$

$$\begin{aligned} \text{Interior angle} &= 180^\circ - \text{exterior angle} \\ &= 180^\circ - 72^\circ \\ &= 108^\circ \end{aligned}$$

$$\text{(c) sum of interior angles} = 5 \times \text{interior angle.}$$

$$\begin{aligned} &= 5 \times 108^\circ \\ &= 540^\circ \end{aligned}$$

$$\text{(d) let the remaining angle be } x, \text{ then}$$

$$\begin{aligned} x + 500^\circ &= 540^\circ \\ x &= 540^\circ - 500^\circ \\ &= 40^\circ \end{aligned}$$

We now state the formulae without proof that:

For a convex polygon, (polygon with no reflex angle) the sum of exterior angles and sum of interior angles can be found by:

- (i) The sum of exterior angles of a convex polygon is 4 right angles. So for any polygon the sum of exterior angles = $4 \times 90^\circ = 360^\circ$.
- (ii) The sum of the interior angles of any n -sided convex polygon is $(2n - 4)$ right angles. This is given as a formula as:
Sum of interior angles = $(2n - 4) \times 90^\circ$.

Note that, a convex polygon is one which does not have a reflex angle among its angles. A reflex angle is an angle which measures more than 180° .

Let us look at another example to apply the formula in (ii) above to find the sum of the interior angles of a polygon.

Example 2

Use the formula to find the sum of the interior angles for a five sided polygon.

Solution

From the above example, we have $n = 5$,

The sum of interior angles = $(2n - 4)$ right angle.

$$\begin{aligned} &= [(2 \times 5) - 4] \times 90^\circ \\ &= (10 - 4) \times 90^\circ \\ &= 6 \times 90^\circ \\ &= 540^\circ \end{aligned}$$

You should now do the last activity so that you assess how well you have understood the angle properties of polygons.

**Activity****Activity 4 The number of sides in a polygon.**

Each interior angle of the regular polygon is 120° .

- (a) Find the number of sides that this polygon has.
- (b) What is the name of this polygon?

Feedback

(a) You may have used the first method as follows:

$$\text{Each exterior angle} = 180^\circ - 120 = 60^\circ.$$

$$\text{exterior angle} = \frac{\text{sum of exterior angle}}{n} = \frac{360^\circ}{n}.$$

$$60^\circ = \frac{360^\circ}{n}$$

$$n = \frac{360^\circ}{60^\circ} = 6.$$

(b) Since the number of sides is 6, the polygon is called a Hexagon.

Or you could have used the formula

(a) Let the polygon have n sides. Hence, the number of interior angles is also n . If each interior angle is 120° then, the sum of interior angles = $120^\circ n$.

But the sum of interior angles is = $(2n - 4)90^\circ$.

$$(2n - 4)90^\circ = 120n^\circ$$

$$180n^\circ - 360^\circ = 120n$$

$$180n^\circ - 120n = 360$$

$$60n^\circ = 360^\circ.$$

$$n = 6.$$

(b) Since it has 6 sides, it is called Hexagon.

You can now read the topic summary to remind yourself on what we have discussed in this Topic 2.

Topic 2 Summary



Summary

You have come to the end of Topic 2 in which you covered the angle properties of triangles and quadrilaterals, the exterior and interior angles of a polygon. You also learned how to find the sum of both the exterior and interior angles. You have learned that;

- Exterior angle = $\frac{360^\circ}{n}$.
- *Exterior angle + Interior angle = 180°.*
- *Sum of exterior angles = 4 × right angles.*
- *Sum of Interior angles = n × Interior angles = (2n - 4)90°*

where n is the number of sides for the polygon.

You should now work out the problems in Topic 2 exercise to assess yourself on how well you have understood the topic. You are advised to revise the sections where you have problems in your topic exercise.

The Unit Summary provided below is meant to remind on the contents of the whole unit. Read it before moving on to the next unit.

Unit Summary



Summary

You have come to the end of unit 10 on the angle properties of a circle and polygons. In this unit, you had two topics, Topic 1 and Topic 2.

In Topic 1, you learnt about the angle properties of a circle. You covered the angles at the centre, angles in the semicircle, angles in the same segment, angles in the opposite segment and tangent properties to the circle.

In Topic 2, you covered the angle properties of triangles and quadrilaterals, the exterior and interior angles in polygons and their sums. You also learned how to find the sum of both the exterior and interior angles. You have learned that;

- Exterior angle = $\frac{360^\circ}{n}$.
- *Exterior angle + Interior angle = 180°.*
- *Sum of exterior angles = 4 × right angles.*
- *Sum of Interior angles = n × Interior angles = (2n - 4)90°*

where n is the number of sides for the polygon.

You were given a number of activities and two End of Topic Exercises. I hope by now you have gone through them, managed to answer all of them and found your progress satisfactory. But if not revise the areas you had difficulties in or take note so that you may present them to your tutor during your tutorial sessions.

There is no end of unit assessment in this unit. You will cover this at the end of unit 12. You will be expected to do a Tutor Marked Assessment (TMA) for the three units starting from unit 10 to 12, as you have done in the previous units.

You may now proceed to the next unit, unit 11 on Mensuration. Mensuration is quite interesting too. I hope you will enjoy studying this next unit.

Reference

Channon JB et al (1985), *New General Mathematics 3*, Longman, Singapore

Mukuyamba A et al (1995), *Mathematics Grade 11 Pupil's Book*, Yorkvik Publishing Limited, Chingola

Assignment



Exercise 1

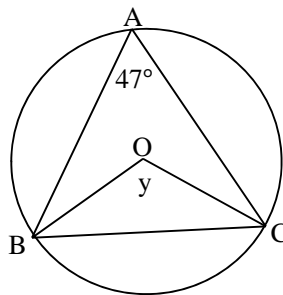
There are two assignments in this unit. These have been expressed as topic exercises. Attempt each after you have studied the section containing the topic.

There are only two questions in Exercise 1. Much of the work will be combined in Exercise 2. The properties learnt are never examined as individual but as a combination in one question. This is what has been expressed in this exercise.

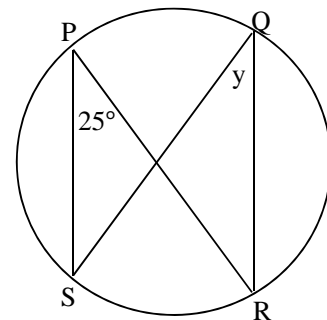
Do not check through the feedback before you attempt the exercise. This will help you appreciate your level of competence of the content that you have studied so far.

Topic 1 Exercise

- Find the angles marked with a letter y in each of the following diagrams:

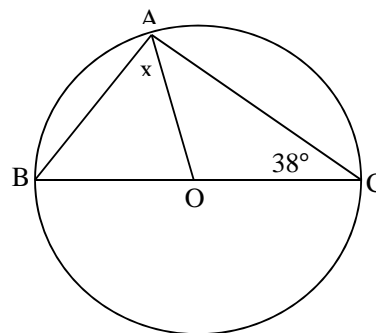


(i)



(ii)

- In the diagram below A, B, C are points on the circle, $AO = OC$, $\widehat{OCA} = 38^\circ$. O is the centre of the circle.



Calculate \widehat{BAO}

Assignment



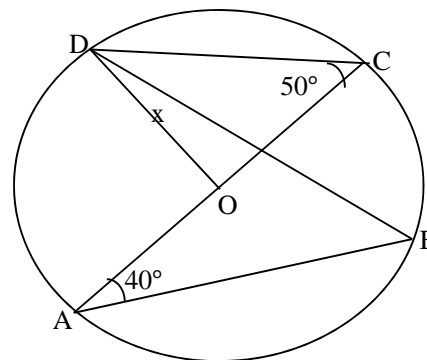
Exercise 2

In this exercise, some of the properties learnt earlier are included. As you attempt these questions, bear in mind that you need to use even the properties from Topic 1.

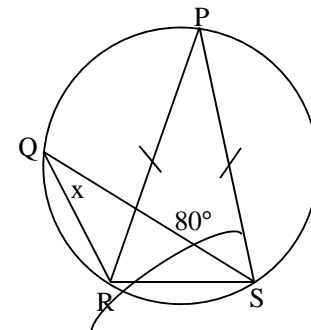
Do not check through the feedback before you attempt the exercise. This will help you appreciate your level of competence of the content that you have studied so far.

Topic 2 Exercise

- Find the angles marked with a letter x in each of the following diagrams:

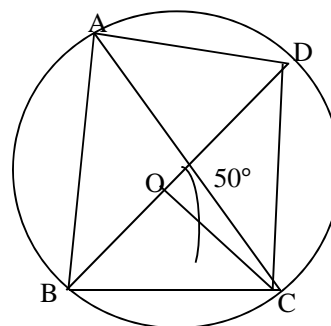


(i)



(ii)

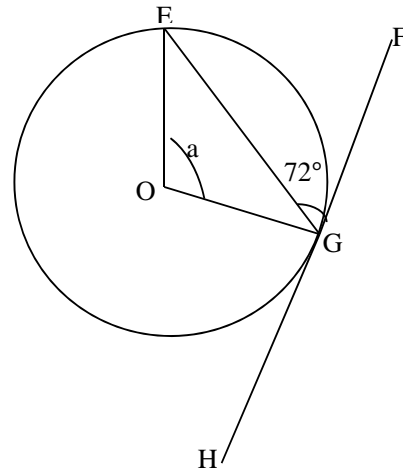
- In the diagram below A, B, C, D are points on the circle, $AD=DC$, $\angle D\hat{O}C = 50^\circ$. O is the centre of the circle.



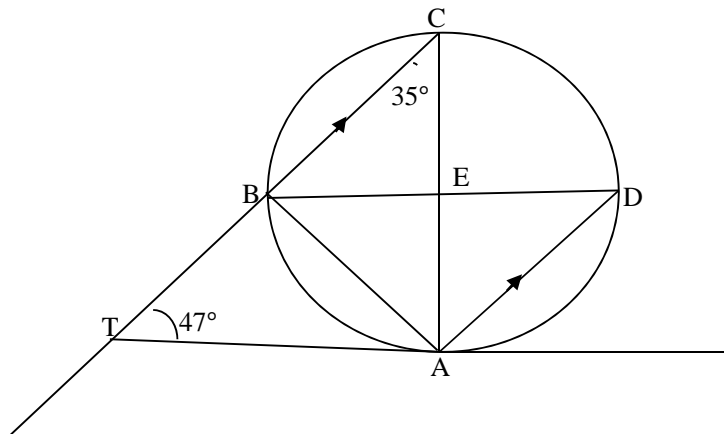
Calculate (i) \widehat{ADC}

(ii) \widehat{ABC} .

3. Find the size of the angle marked by the letter a .



4. In the diagram, TBC is parallel to AD , $\widehat{ACB} = 35^\circ$, $\widehat{ATB} = 47^\circ$.



Calculate:

(i) \widehat{ABT}

(ii) \widehat{AEB}

END

Answers to Exercises



Feedback

When you have done the work above, you should compare your answers with these below. Feel free to use any other method you may have come across.

Topic 1 Exercise

1. (i) $\widehat{BAC} = 47^\circ$, $\widehat{BOC} = y$

Property 1: Angle at the circumference is half the angle at the centre by the same chord

$$Y = 2 \times \widehat{BAC}$$

$$Y = 2 \times 47^\circ$$

$$Y = 94^\circ$$

Therefore, $\widehat{BOC} = 94^\circ$

(ii) Property 2: Angles in the same segment are equal

$$\widehat{SPR} = \widehat{SQR}$$

Since $\widehat{SPR} = 25$

Therefore, $\widehat{SQR} = 25$

2. Property 3: Angle in a semi-circle is 90°

$$\widehat{BAC} = 90^\circ$$

Since $AO = OC$, triangle AOC is an Isosceles Triangle.

Hence, $\widehat{OCA} = \widehat{OAC} = 38^\circ$ (Base angles in Isosceles)

It follows that, $\widehat{OAC} = 38^\circ$

We know that $\widehat{BAC} = 90^\circ$

$$\text{But } \widehat{BAC} = \widehat{BAO} + \widehat{OAC}$$

$$\widehat{BAO} + \widehat{OAC} = \widehat{BAC}$$

$$\widehat{BAO} + 38^\circ = 90^\circ$$

$$\widehat{BAO} = 90^\circ - 38^\circ$$

$$\widehat{BAO} = 52^\circ$$

Therefore, $y = 52^\circ$.

Topic 2 Exercise

1. (i) Property 1: angle at the centre is twice angle at circumference

$$\widehat{DOA} = 2\widehat{DCA}$$

$$\widehat{DOA} = 2 \times 50^\circ$$

$$\widehat{DOA} = 100^\circ$$

Property 2: angle in same segment are equal

$$\widehat{BAC} = \widehat{BDC} = 40^\circ$$

$$\widehat{BDC} = 40^\circ$$

Property: Base angles of isosceles triangle are equal

$$\widehat{ODC} = \widehat{OCD} = 50^\circ$$

$$\text{But } \widehat{BDC} = 40^\circ$$

$$\widehat{ODB} + \widehat{BDC} = \widehat{ODC}$$

$$x + 40^\circ = 50^\circ$$

$$x = 50^\circ - 40^\circ$$

$$x = 10^\circ$$

Therefore, $\widehat{ODB} = 10^\circ$

(ii) Property: angles in the same segment are equal

$$\widehat{RQS} = \widehat{RPS}$$

Property: Base angles in an Isosceles triangle are equal

$$\widehat{PRS} = \widehat{PSR} = 80^\circ$$

Property: Sum of angles in a triangle = 180°

$$\widehat{PRS} + \widehat{PSR} + \widehat{RPS} = 180^\circ$$

$$80^\circ + 80^\circ + \widehat{RPS} = 180^\circ$$

$$160^\circ + \widehat{RPS} = 180^\circ$$

$$\widehat{RPS} = 180^\circ - 160^\circ$$

$$\widehat{RPS} = 20^\circ$$

Since $\widehat{RPS} = 20^\circ$, then $\widehat{RQS} = 20^\circ$

2. BD is a straight line.

Property: Angles on a straight line add up to 180°

$$\widehat{BOC} + \widehat{COD} = 180^\circ$$

$$\widehat{BOC} + 50^\circ = 180^\circ$$

$$\widehat{BOC} = 180^\circ - 50^\circ$$

$$\widehat{BOC} = 130^\circ$$

Property: Angle at the centre is twice angle at the circumference

$$\widehat{BOC} = 2 \times \widehat{BAC}$$

$$130^\circ = 2 \times \widehat{BAC}$$

$$\widehat{BAC} = \frac{130^\circ}{2}$$

$$\widehat{BAC} = 65^\circ$$

Property: Angles in the same segment are equal

$$\widehat{BAC} = \widehat{BDC}$$

$$\widehat{BDC} = 65^\circ$$

Property: Base angles of an Isosceles triangle are equal

$$\widehat{OBC} = \widehat{OCB} = 25^\circ$$

Property: angles in the same segment are equal

$$\widehat{CBD} = \widehat{CAD} = 25^\circ$$

$$\widehat{CAD} = 25^\circ$$

Base angles of an Isosceles triangle are equal

$$\widehat{CAD} = \widehat{DCA} = 25^\circ$$

$$\widehat{DCA} = 25^\circ$$

$$(i) \quad \widehat{ADC} = \widehat{ADB} + \widehat{BDC}$$

$$\widehat{ADC} = 65^\circ + 65^\circ$$

$$\widehat{ADC} = 130^\circ$$

$$(ii) \quad \widehat{ABC} = \widehat{CBD} + \widehat{DBA}$$

$$\widehat{ABC} = 25^\circ + 25^\circ$$

$$\widehat{ABC} = 50^\circ$$

3. Property: Angle formed by a tangent to a circle with the radius.

$$\widehat{OGF} = 90^\circ$$

$$\widehat{OGE} + \widehat{EGF} = 90^\circ$$

$$\widehat{OGE} + 72^\circ = 90^\circ$$

$$\widehat{OGE} = 90^\circ - 72^\circ$$

$$\widehat{OGE} = 18^\circ$$

$$OG = OE \text{ (radii)}$$

Triangle EOG is Isosceles

$$\widehat{OGE} = \widehat{GEO} = 18^\circ \text{ [Base angles]}$$

$$a + \widehat{OGE} + \widehat{GEO} = 180^\circ \text{ [sum of angles in a triangle]}$$

$$a + 18^\circ + 18^\circ = 180^\circ$$

$$a + 36^\circ = 180^\circ$$

$$a = 180^\circ - 36^\circ$$

$$a = 144^\circ$$

4. Property: Angles in alternate segment

$$\widehat{TAB} = \widehat{ACB}$$

$$\widehat{TAB} = 35$$

Property: Angles in same segment are equal

$$\widehat{BCA} = \widehat{BDA} = 35$$

$$\widehat{BDA} = 35$$

Property: Alternate angles in parallel lines

$$\widehat{ADB} = \widehat{DBC} = 35^\circ$$

$$\widehat{DBC} = 35^\circ$$

Property: Straight line angles add up to 180°

$$\widehat{TBA} + \widehat{ABC} = 180^\circ$$

- (i) Property: Sum of two opposite interior angles is equal to the exterior angle.

$$\widehat{BTA} + \widehat{TAB} = \widehat{ABC}$$

$$47^\circ + 35^\circ = \widehat{ABC}$$

$$\widehat{ABC} = 82^\circ$$

- (ii) Same property as in (i) above.

$$\widehat{EBC} + \widehat{BCE} = \widehat{BEA}$$

$$35^\circ + 35^\circ = \widehat{BEA}$$

$$70^\circ = \widehat{BEA}$$

$$\widehat{BEA} = 70^\circ$$

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Unit 13

Trigonometry

Introduction

You have studied 12 units in this course so far. The knowledge you have is not to be thrown away but used in answering questions in mathematics and solving life problems. You learn mathematics not only to enable you to write a mathematics examination and pass but to use mathematics concepts in life.

This unit is about trigonometry and will have two topics. Although some terms may be new, you will realise that you have come across some of the content before; for example, when you studied angles in your Junior Certificate Program and in mathematics 10. You learnt to define an angle and to state the different size of an angle and the names such as acute angle, obtuse angle, and reflex angle. You also discussed different shapes such as polygons. In polygons, you learnt about triangles, rectangles, and the other shapes. You learnt about three figure bearings under scale drawing and bearings in Grade 10. All this knowledge will be very useful for our discussions in this unit.

This unit will, thus, build on your previous knowledge. We will discuss the relationship of angles in a triangle to the sides of the triangle. You will learn that the relationship between the sides of the same triangle give rise to an angle in the triangle defined by the two given sides.

In each topic, you will be required to do some activities and a topic exercise. There is also a self-assessment activity at the end. It should not be mistaken for a tutor marked assignment (TMA). You are encouraged to answer all the questions in the activities and topic exercises. In this unit, we have used the same icons you have come across before. It is important that you revise the icons for you to master their usage in the unit.

By the end of the unit, you should be able to achieve the following outcomes:



- *Find* the trigonometric ratios of sine, cosine and tangent of an acute angle.
- *Solve* problems involving right-angled triangles using sine, cosine and tangent ratios.
- *Apply* trigonometric ratios to calculate angles, height and distances.
- *Apply* trigonometric ratios to solve problems involving three figure bearing.

Time Frame



It is estimated that to complete studying this unit you will need between 10 to 12 hours. Since there are two topics, you are expected to take 7 hours on topic one and the remaining 5 hours on topic two. Do not worry if you take longer in finishing this unit. Remember that we all have different abilities and hence we study at different paces.

You are encouraged to spend 2 hours answering each topic exercise in this unit. Since there are two topic exercises in this unit, this means that you will spend 4 hours on these exercises.

You will be required to spend 2 hours on the self-assessment activity.

The total hours for completing the unit will thus be between 16 and 18 hours.

Learning Resources



You will need the following materials:

- Ruler
- Pencil
- Pen
- Scientific calculator or
- Mathematical Tables

Teaching and Learning Approaches



During your study of this unit, understanding the concept of trigonometry ratios is of great importance. You will achieve this understanding of trigonometry by working out problems involving trigonometry ratios. You need to do a lot of practice. Through your interpretation of the relationship between sides and angles, you will appreciate the concept of trigonometry. Mastery of trigonometry skills comes as you practice working out the activities and exercises as an individual or in a group with others studying the same unit.

You will come across spaces left in the unit during your study. The spaces are meant for you to use as you interact with the materials. You need to use them work out certain questions in the activities given to you. At other times, you will be asked to provide your own plain sheets of paper to work on. Self-assessment is very necessary. Self-assessment is done through the activities and exercises you will be working out as you study. You will be assessing your knowledge and skills acquired in the process of studying. Mark your own work using the feedback provided immediately. It is expected of you not to go through the feedback before doing the activity.

Terminology



Deriving:	To form or generate something, may be a formula.
Extraction:	A copy of something especially from books, magazines and so on.
Inverse:	Refers to reversing the effect of an operation.
Sketching:	Free-hand drawing of an object.

Topic 1 Trigonometry Ratios

The term ‘trigonometry’ may be new to you however, as stated in the introduction section; you already have previous knowledge about angles from your Junior Certificate program and your grade 10 work. In this topic you will learn to define a trigonometry ratio. You will also learn how to find the three ratios. You will further learn how to use a scientific calculator to find the values of the ratios or the angle of the ratio.

You learnt about ratios in unit 5 of your grade 11 work. You looked at how to express quantities in ratios and how to use ratios to solve problems. You will use this knowledge of ratios in this unit. The content may not be exactly as you learnt previously but the principle of ratios will be applied. Remember that in unit 5, we defined a ratio as the comparison of two or more quantities. In our discussion of trigonometry ratios, we will be comparing sides of the same triangle.

After studying the topic you will be required to answer a self-assessment activity at the end of the topic. This is in addition to the two topic exercises in the unit. As usual, we would like to encourage you to work on the activities and topic exercises first before looking at the feedback.



In this topic, we will address the first two of the four unit outcomes:

- *Find* the trigonometric ratios of sine, cosine and tangent of acute angle.
- *Solve* problems involving right-angled triangles using sine, cosine and tangent ratios.

To achieve the above stated outcomes, you should be able to realize the following objectives at the end of the topic:



- *Define* trigonometric ratios.
- *Use* the mathematical tables or scientific calculator to find the angle or ratios.
- *Find* the trigonometric ratios of sine, cosine and tangent of acute angle.
- *Solve* problems involving right-angled triangles using sine, cosine and tangent ratios.

Trigonometry Ratios

Defining Trigonometric Ratios

In this section, we will work to define trigonometric ratios. But before that, let us examine the meaning of trigonometry. According to Larson E R and et al (1998), the term **TRIGONOMETRY** comes from the Greek words, *trigonon* which means *triangle* and *metron* which means *measure* respectively.

From the above, we note that TRIGONOMETRY is a branch of mathematics that deals with measuring triangles, particularly the plane triangles. It deals with the relationships between the sides and the angles of any given triangle.

In the next situation, we will work out the relationship explained above. We will consider the relationship between the angles and the sides of the same triangle.

Let us consider the right-angled triangle ABC shown in Figure 1 below:

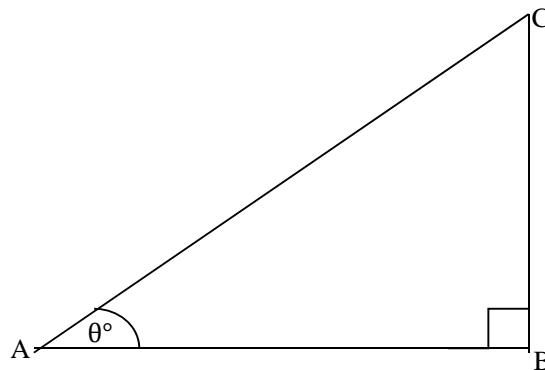


Figure 1: A right-angled triangle.

We are now going to study the sides and angles in the triangles above. We have to see the relationship of these two, i.e. sides and angles, and from that information we will make conclusions about the relationship. This will help us define the trigonometry ratios.

Naming the Sides

We are first going to name the sides of this triangle in relation to the angle given within the triangle.



Let us complete the following activity. This activity will help you remember the knowledge learnt for the pythagoras theorem in grade 9 mathematics class.

Do you remember the name of the longest side of the right angled triangle? Write your answer in the space below.

*We trust that you still remembered the Pythagoras Theorem. We hope that your answer was that the longest side is called the **hypotenuse**.*

The above answer means that the side (in Figure 1) AC is the **hypotenuse** (hyp.) since it is the longest side. The side BC, which is opposite to the angle θ is called the **opposite side** (Opp.). The other remaining side AB is called the **adjacent side** (Adj.).

The symbol θ is a Greek letter from the Greek alphabet. In mathematics, the symbol is used to represent an angle. This is the common symbol used for an angle. It is read as **theta**.

The names of the sides are shown in the Figure 1.1 below

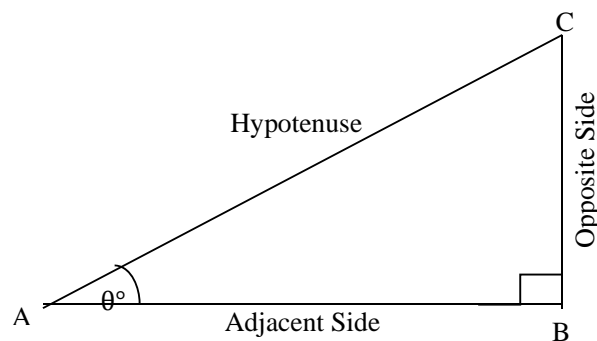


Figure 1.1: Names of the sides of a right-angled triangle

We now are, by way of an example, going to consider the right-angled triangle in Figure 1.2 below:

Given that $AB = 4$ cm, $BC = 5$ cm, $AC = 3$ cm. Write down the measurements of the sides in the spaces provided below the triangle:

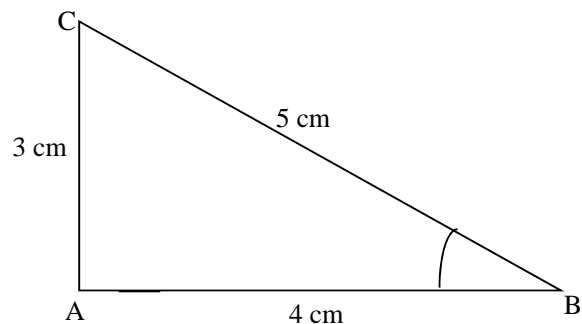


Figure 1.2: Sides and lengths of a triangle

Hypotenuse side.....

Opposite side.....

Adjacent side.....

Solution

The measurements of the sides given are:

- (i) Hypotenuse is 5 cm
- (ii) Adjacent side is 4 cm
- (iii) Opposite side is 3 cm.

We will come back to use this triangle in figure 1.2 later on in the unit. The example was meant as practice for connecting the names of the sides of the triangle and the measurements assigned to each side. It is important to identify the correct measurement for a particular side as these values give us correct ratio values.

We have looked at the names of the sides of a triangle, particularly a right-angled triangle. We have also looked at the sides and their related measurements. We now move on to identify the ratios of trigonometry in any given triangle.

Identifying Trigonometry Ratios

We are now going to use the right-angled triangle below, to illustrate that the ratio of sides of the triangle gives the relationship of the sides to the angle in the triangle.

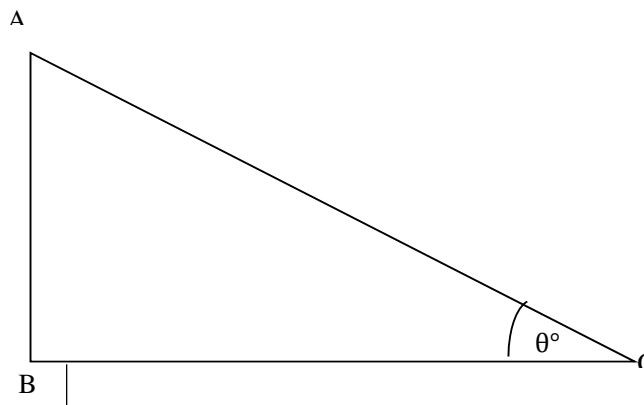


Figure 1.3: A right-angled triangle showing the angle, θ .

From your work on ratios in the unit on ratio, proportion and rate, you learnt that a ratio is written as $\frac{a}{b}$. This is the format we will use in writing our ratios.

From the above triangle, we have the following ratios and we will name the ratios X, Y and Z as shown:

$$\frac{AB}{AC} = \text{ratio X}$$

$$\frac{CB}{AC} = \text{ratio Y}$$

$$\frac{AB}{BC} = \text{ratio Z}$$

The value of the ratio depends on the value of the angle given. In this case, we should realise that each of these ratios stated above, depend on the value of angle θ° . This means that each angle given in a right-angled triangle, gives a ratio which relates two sides of the triangle to the given angle. This relationship is what we are studying in this unit.



ALL the ratios are in relation to the angle θ° .

Deriving the Trigonometry ratios

Having dealt with the ratios, we are now going to identify which particular sides give a particular ratio. To do this, we will still use the ratios above and the sides of the triangle in figure 1.3. We will discuss each ratio at a time.

Now, let us start with the first ratio.

Using the letters of the sides of the triangle, we have:

$$\text{Ratio X} = \frac{AB}{AC}$$

You will notice that this is the ratio of the Opposite side to the Hypotenuse side of the triangle. This is written as:

$$\text{Ratio X} = \frac{\text{Opp. side of triangle ABC}}{\text{Hypotenuse of triangle ABC}}$$

This type of ratio is called the **sine of θ°** . This may also be abbreviated to **Sin θ°** .

Hence, Ratio X = Sin θ°

$$\text{Sin } \theta^\circ = \frac{AB}{AC}$$

Therefore, the sine of angle θ is the ratio of the side opposite to the hypotenuse side of the triangle.

$$\text{Sin } \theta^\circ = \frac{\text{Opposite side to the angle}}{\text{Hypotenuse side}} = \frac{\text{Opp.}}{\text{Hyp.}}$$

We have now identified which sides give the ratio of sine. We have concluded that the opposite side divided by the hypotenuse side gives the ratio of sine.

Let us now identify the sides giving the other ratio.

We move on to the next ratio:

Using the sides of the triangle, we have:

$$\text{Ratio Y} = \frac{CB}{AC}$$

You will notice also here that this is the ratio of the adjacent side to the Hypotenuse side of the triangle.

$$\text{Ratio Y} = \frac{\text{Adj. of triangle ABC}}{\text{Hypotenuse of triangle ABC}}$$

This type of ratio is called the **cosine of θ°** . This is abbreviated to **Cos θ°** .

Hence, Ratio Y = Cos θ°

$$\text{Cos } \theta^\circ = \frac{CB}{AC}$$

Therefore, the cosine of angle θ is the ratio of the adjacent side to the hypotenuse side.

$$\text{Cos } \theta^\circ = \frac{\text{Adjacent side to the angle}}{\text{Hypotenuse side}} = \frac{\text{Adj.}}{\text{Hyp.}}$$

We have now identified which sides give the ratio of cosine. We have concluded that the adjacent side divided by the hypotenuse side gives the ratio of cosine.

Let us identify the sides that give the last ratio in our discussion.

Now, move on to the last ratio:

Using the sides of the triangle, we have:

$$\text{Ratio Z} = \frac{AB}{BC}$$

You will notice that this is the ratio of the Opposite side to the adjacent side of the triangle.

$$\text{Ratio Z} = \frac{\text{Opp.of triangle ABC}}{\text{Adj.of triangle ABC}}$$

This type of ratio is called the **tangent of θ°** . This is abbreviated to **$\tan \theta^\circ$** .

Hence, Ratio Z = $\tan \theta^\circ$

$$\tan \theta^\circ = \frac{AB}{BC}$$

Therefore, the tangent of angle θ is the ratio of the opposite side to the adjacent side.

$$\text{Tan } \theta^\circ = \frac{\text{Opp.side of triangle ABC}}{\text{Adj. side of triangle ABC}} = \frac{\text{Opp.}}{\text{Adj.}}$$

We have seen that the opposite side and the adjacent side give the ratio of the tangent.

We have seen how the ratios of the trigonometry are derived from the sides of a Right-angled triangle. It is important for you to understand and master this information. As we now progress to deal with the application of the trigonometry ratios in mathematics, this information will be very useful.

Many people find it easy to remember what sides of the right-angled triangle are to be used in relation to sine, cosine or tangent by memorizing the: **SOH-CAH-TOA**. The acronym stands as follows:

$$\text{SOH stand for Sine of } \theta = \frac{\text{opposite Side}}{\text{Hypotenuse side}}$$

$$\text{CAH stands for Cosine of } \theta = \frac{\text{Adjacent Side}}{\text{hypotenuse side}}$$

$$\text{TOA stands for Tangent of } \theta = \frac{\text{opposite Side}}{\text{Adjacent Side}}$$

We would like to encourage you to learn the acronym to remember it and use it in any situation.

We have come to the end of this part on deriving the trigonometry ratios. We will now look at how to find the ratio values using the mathematical tables or the scientific calculator.

Finding the Angle values of Sine, Cosine and Tangent Ratios

In the Zambian situation, there was a time when in schools not everyone had access to a scientific calculator. Most schools, if not all, had no access to scientific calculator. This prompted the use of mathematical tables in answering such kind of questions in mathematics examinations. However, it should be noted that today the scientific calculator has been made very much available on the market. Accessing the calculator, however, is an individual issue which depends on the individual income or financial support.

The design of the mathematical table is such that it operates as a scientific calculator. While a scientific calculator can take any value, the mathematical table only take up values in angles from 0° to 89.9° . Any value above 89.9° is found by associating such a value to the values within 0° to 89.9° .

With this in mind, it means then that an extraction of the mathematical tables will be used to show how to get these angles in situations where a scientific calculator is missing. The procedures will be shown in detail as to how to go about the whole issue of calculations. At the end of the day, we expect that you should be able to use a mathematical table or a scientific calculator to solve problems involving trigonometry.

The values of sine ratio, cosine ratio and tangent ratio for angles from 0° to 89.9° are given in mathematical tables under each heading, if you are using the mathematical tables. If you are using a scientific calculator, you need not worry as these values are given as you key in angle values for the particular trigonometry ratio.

Let us consider the following example to illustrate this point. We will use the triangle in figure 1.2 above. Remember that we said we will go back to use the triangle.

This example will help you in practicing using the mathematical tables. In this work, we will work so much with mathematical tables as this gives many students challenges in finding the values of the ratios and the angles in the absence of a scientific calculator

Example 1

In the triangle below, $AB = 4$ cm, $AC = 3$ cm and $BC = 5$ cm.

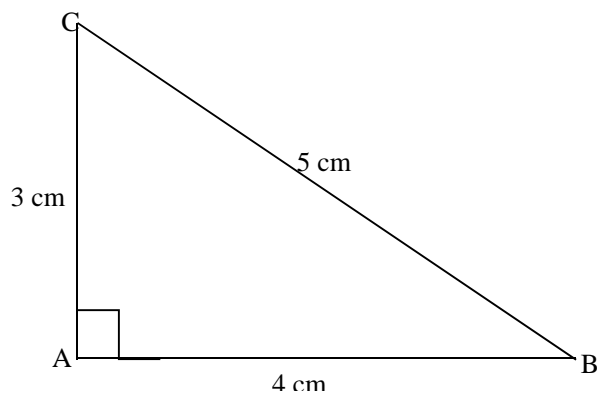


Figure 1.4: Showing a triangle with all sides given.

Solution

You should note that angle A is 90° . (Right-angled)

Angle B can be found by using sine ratio.

$$\sin B = \frac{\text{Opp.side}}{\text{Hypotenuse}}$$

From the triangle, opposite side = 3 cm and hypotenuse side = 5 cm.

$$\text{Therefore, } \sin B = \frac{3}{5}$$

$$\sin B = 0.6000 \text{ (to 4 Sig. figures)}$$

Sin B is read as the sine ratio of angle B.

So to express B as an angle, we find the inverse of sine ratio. The word, inverse in mathematics is used to refer to the reverse process.

$$B = \sin^{-1} 0.6000 \text{ (read as 'the sine inverse' of 0.6000)}$$

To find the angle which gives the sine ratio of 0.6000, we use the natural sine page or sine of angles page from the mathematical table as shown below. You can also key in the value into a scientific calculator directly and press the sine key.

When you use the mathematical table, we follow the procedure shown below and explained just below the table.

Sines of angles

θ	0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	ADD Differences				
	1'	2'	3'	4'	5'	6'	7'	8'	9'	10'	1'	2'	3'	4'	5'
0°	0.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0.0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0.0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	0.0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	0.0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	15
5	0.0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	0.1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	0.1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	0.1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	0.1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10°	0.1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	11	14
11	0.1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	0.2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	0.2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	0.2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	0.2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	0.2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	0.2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	0.3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	0.3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20°	0.3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	0.3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	0.3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	0.3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	0.4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	0.4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	0.4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	0.4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	0.4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	0.4848	4863	4879	4893	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30°	0.5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	0.5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	0.5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	0.5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	0.5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	0.5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	0.5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	0.6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	0.6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	0.6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40°	0.6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	0.6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	0.6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	0.6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	0.6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

Table 1: Extract of Sines of angles showing how to get ratio 0.6000



The ratios of angles of sine are given in the main table as above.

The bold numbers on the far left side are the angles and their fractions on the top part of the table. These are measured in degrees and or minutes. The remainders or differences are reflected in the left side column and the first row above. The dots mean that the figures (numbers) continue on until 90°.

We should note that we do not just use any table found at the back of the module or text book. We use the sine table when dealing with sine ratios.

For us to find the angle, we begin inside the main table where the ratios are. We have to get 0.6000. If we fail to get the 0.6000

among the ratios, we get the next nearest ratio to our value of 0.6000.

In the table above, the nearest ratios are 5990 and 6004.

In the table on the sines of angles, the two ratios are shown by the two lines drawn from above the table meeting the line from the angle.



The ratios we picked to use were written as 5990 and 6004. These are not whole numbers. These ratios are considered as 0.5990 and 0.6004. The ratio 5990 is before 6000 and 6004 is after 6000.

To understand how to get the angles represented by these ratios, we will use two procedures. You should note that the procedures are but ways of coming up with some answers. The first one deals with a situation where the ratio picked from the table is before the required value. The second procedure is when the ratio picked is after the required value. **We should note that for anyone using a scientific calculator, the two procedures do not apply.**

Let us now start our discussion on the first procedure.

PROCEDURE A

We are using 0.5990, which is before 0.6000,

We first read out from the table the angle value of 0.5990. The angle value of this ratio is $36^\circ 48'$. Remember that this angle is not what we are looking for. We are looking for the angle for 0.6000 but we have got the angle for 0.5990.

We then have to find the difference between 0.5990 and our 0.6000. This difference is to help us find the extra minutes which we either add to $36^\circ 48'$ to get our angle or subtract from $36^\circ 48'$ to get our angle.

$$\text{Difference} = 0.6000 - 0.5990$$

$$\text{Difference} = 10$$



The differences are never written as decimal number. However, they are treated as decimal differences. Please take care as you deal with the differences.

Along the line where this number (**0.5990**) is located, we need to find this difference (**10**) in the last column. If we can't, we have to find the closest number. In the column, it is **9**.

9 is directly below 4' (read as 4 minutes). We then add this 4' to $36^\circ 48'$.

Therefore, angle B = $36^\circ 48' + 4'$

$$\text{Angle B} = \mathbf{36^\circ 52'}$$

Let us now look at procedure 2.

PROCEDURE B

We are using 0.6004, which is after 6000

We get the angle $36^\circ 54'$ for the ratio 0.6004. As stated above, this is not the required angle. We have to find the difference that will help us find the required angle:

$$\text{Difference} = 0.6004 - 0.6000$$

$$\text{Difference} = 4$$

Along the line where this number (**0.6004**) is located, we need to find this difference in the last column. Again here if we cannot, we find the closest. In the column, **4** is present and is directly below 2'. (Read as 2 minutes)

Therefore, angle B = $36^\circ 54' - 2'$

$$\text{Angle B} = \mathbf{36^\circ 52'}$$

Since 52' is very close to 54' in the table, we can round of 52' as 0.9° . Therefore, the final answer correct to 1 decimal place is:

$$\text{Angle B} = \mathbf{36.9^\circ}$$

We have shown how to use the procedures and come up with the angle required. You should have noted that whichever procedure you use, the answer will be the same or very close to each other. When there is a difference in the answers between the two procedures, the difference is very minimal. Many times there will be no difference as seen in our case above.

The procedures shown and used above can be used for finding the angles or ratios under the cosine and tangent by following the same steps.

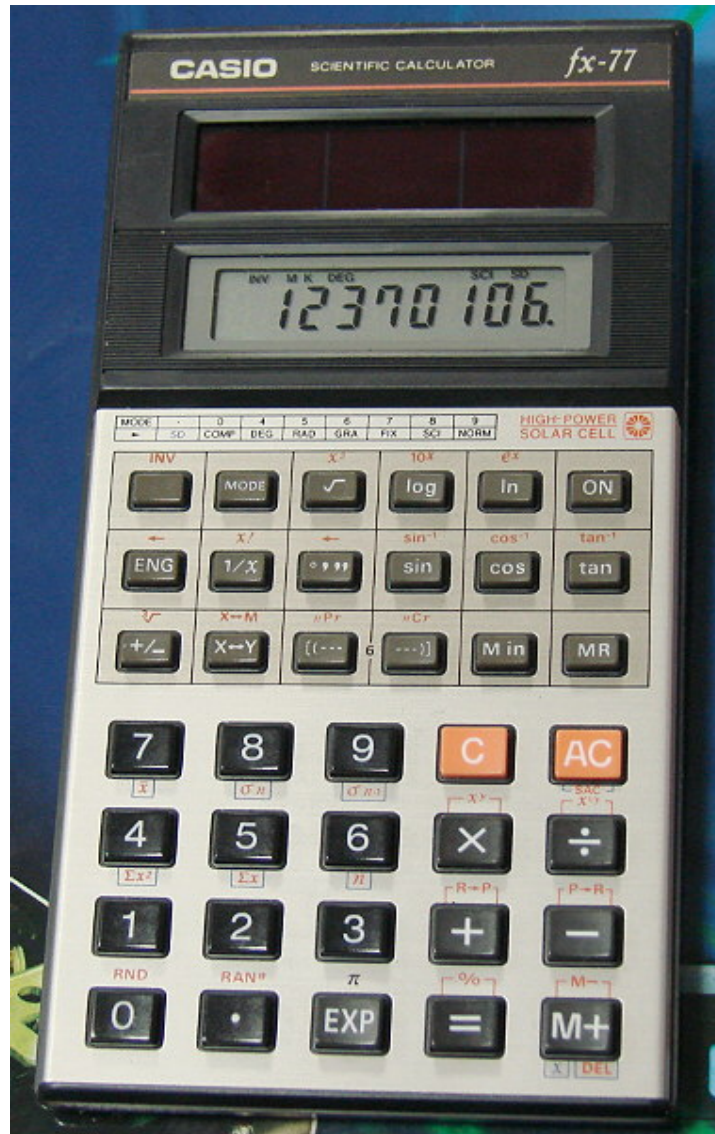
The only difference lies in the fact that the table pages used will be different. You should note that each ratio has its own table. This means that whenever you are working with trigonometry ratios and you are using a mathematical table, you need to pay attention to use the individual ratio table.

This part was dedicated to the process of using the mathematical tables in the absence of a scientific calculator. What we have basically done, is to show how to use the mathematical tables in the calculation of the ratios or the angles under trigonometry.

Let us now show how to use the scientific calculator. This is being done on the basis that there are some students that may have access to a scientific calculator.

A scientific calculator will appear like this calculator in the picture below. Apart from the numeric key pad (the number keys) which every other ordinary calculator has, a scientific calculator has a scientific key pad. This has extra keys used in engineering circles. This makes the calculation of mathematics problems at a higher level possible.

Scientific calculator



Picture1: Solar powered Scientific Calculator

We will now consider working with a scientific calculator to find the value of the angle B given the sine ratio as shown:

$$\sin B = 0.6000$$

Using a scientific calculator, the work is made very easy. We just key in 0.6000 and press the Sin key on the key pad as shown below.

The figure below shows you the procedure of keying in the numbers in the calculator. You first key in 3 divide by 5. You then

key in the second function key followed by the Sin key to get the angle 36.86989...

$$3 \div 5 = 0.6000 \quad \text{2ndF} \quad \text{Sin} \quad 36.86989..$$

The final answer is rounded off to 3 significant figures. This means that you round off the answer to 1 decimal point. Therefore, the answer becomes: $B = 36.9^\circ$.



Let us consider the following activity. This activity you have to work out before comparing your work with that --shown below in the feedback. It will help you practice calculating angles.

Activity 1

Calculate the angle shown in the triangle.

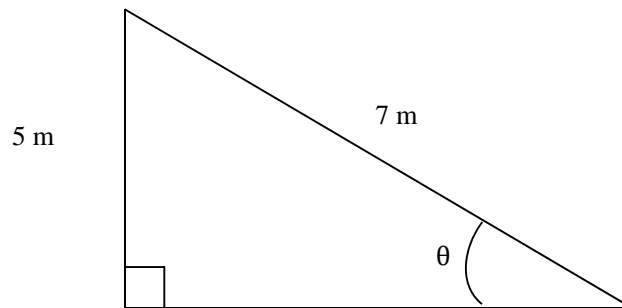


Figure 1.5: Showing a triangle given two sides



Here is the feedback for the work above. You should check it through after you have done the activity. The choice of using either a mathematical table or a scientific calculator is left in your hands. We have shown how to use these instruments in the calculation of mathematical problems involving trigonometry. The feedback below is given using both instruments to show how this should have been done.

Feedback

From the diagram, we notice that this is a right-angled triangle. The hypotenuse side is given and the opposite side to the angle. This means that we are to use the Sine ratio.

$$\sin \theta^\circ = \frac{\text{opp}}{\text{hyp.}}$$

$$\sin \theta^\circ = \frac{5}{7}$$

$$\sin \theta^\circ = 0.7143$$

Please note that if you do not have a scientific calculator, you may use an ordinary calculator to get the quotient of 5 and 7 and express it to four decimal places.

Using the mathematical table, we choose to use **procedure B**. You should note here that which procedure we use is entirely up to an individual.

Since we are dealing with the sine ratio, we will use the sine ratio table provided at the end of this unit.

The ratio 0.7143 is not in the table but 0.7145 is there. We read out the angle of this ratio first. The angle is $45^\circ 36'$

Then we find the difference between the two numbers

$$\text{Difference} = 0.7145 - 0.7143$$

$$\text{Difference} = 2$$

The difference 2 is directly below 1' in the last column.

$$\text{Therefore, angle } \theta = 45^\circ 36' - 1'$$

$$\text{Angle } \theta = 45^\circ 35'$$

$$\text{Therefore, } \theta = 45^\circ 35'$$

To write this answer correct to 1 decimal place, we consider 35' in the column of ratios.

35' is very close to 36' which is 0.6° .

$$\text{Therefore, the angle } \theta = 45.6^\circ$$

Scientific Calculator

When we use the scientific calculator, we should note that we just key in the numbers as shown below and we have the following:

$$5 \div 7 = 0.7143 \quad \text{2ndF} \quad \text{Sin} \quad 45.58469..$$

The final answer is rounded off to 3 significant figures. This means that you round off the answer to 1 decimal point. Therefore, the answer becomes: $\theta = 45.6^\circ$.



You will do two more activities. These activities help you work with the ratios were you make a choice what to use. One will be on cosine ratio and the other will be on tangent ratio. The choice of the instrument to use is up to you as an individual.

We allow you to work out each one of the activities before you compare your work with those provided in the feedback.

Activity 2

This activity will be based on the triangle shown below.

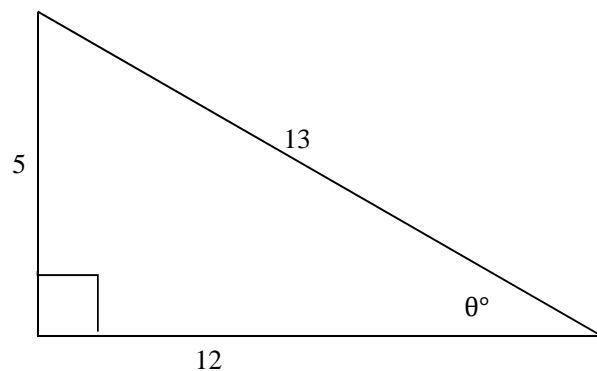


Figure 1.6: Showing a right-angled triangle with an angle of θ°



Here is the feedback for the work above. You should check it through after you have done the activity. In most of these activities, you are to choose the instrument to use. You will notice that most of the work on the procedures is left out. We hope that you have gained enough knowledge and skill to handle the following activities. Again it should be stated that whichever instrument is used is no longer important. The important thing is that the answer should be found.

Feedback

We are going to have to use the triangle shown above. In this triangle:

$$\text{Cos } \theta^\circ = \frac{\text{Adj.}}{\text{Hyp.}}$$

We are to substitute for the Adjacent and Hypotenuse side.

$$\text{Cos } \theta^\circ = \frac{12}{13}$$

$$\text{Cos } \theta^\circ = 0.9231$$

$$\theta^\circ = \cos^{-1} 0.9231$$

Using the table of cosines,

Cosines of angles

θ											SUBTRACT Differences				
	0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1'	2'	3'	4'	5'
0	1.000	1.000	1.000	1.000	1.000	1.000	.9999	.9999	.9999	.9999	0	0	0	0	0
1	0.9998	9998	9998	9997	9997	9997	9996	9996	9995	9995	0	0	0	0	0
2	0.9994	9993	9993	9992	9992	9992	9991	9991	9990	9990	0	0	0	1	1
3	0.9988	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	0.9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	0.9962	9960	9959	9957	9956	9954	9952	9951	9949	9947	0	1	1	1	2
6	0.9943	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	0.9923	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	0.9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	0.9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10	0.9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	0.9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	0.9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	2	3
13	0.9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	2	3
14	0.9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	2	3
15	0.9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	2	3
16	0.9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	2	3
17	0.9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	2	2	3
18	0.9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	2	2	3
19	0.9453	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	2	2	3
20	0.9397	9391	9385	9379	9373	9367	9361	9354	9348	9342	1	2	2	2	3
21	0.9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	2	2	3
22	0.9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	2	2	3
23	0.9203	9196	9191	9184	9178	9171	9164	9157	9150	9143	1	2	2	2	3
24	0.9143	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	2	2	3
25	0.9083	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	2	2	2	3
26	0.8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	2	2	2	3
27	0.8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	2	2	2	3
28	0.8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	2	2	2	3
29	0.8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	2	2	2	3
30	0.8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	2	2	2	3
31	0.8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	2	2	2	3
32	0.8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	2	2	2	3
33	0.8385	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	2	2	2	3
34	0.8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	2	2	2	3
35	0.8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	2	2	2	3
36	0.8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	2	2	2	3
37	0.7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	2	2	2	3
38	0.7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	2	2	2	3
39	0.7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	2	2	2	3
40	0.7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	2	2	2	3
41	0.7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	2	2	2	3
42	0.7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	2	2	2	3
43	0.7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	2	2	2	3
44	0.7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	2	2	2	3

Table 2: Extract from the Cosines of angles showing how to get the ratio 0.9231

We are using the mathematical table. We can use any of the procedures worked out in the pages above:

Our ratio is 0.9231.

Using Procedure A, we will take 0.9225 which is the closest ratio to our ratio. The angle for this ratio (0.9225) is $22^\circ 42'$.

We then find the difference between the numbers

$$\text{Difference} = 0.9231 - 0.9225$$

$$\text{Difference} = 6$$

This difference 6 is directly below 5' in the last column. This difference is then subtracted as shown below:

$$\text{Then, } \theta^\circ = 22^\circ 42' - 5'$$

$$\text{Therefore, } \theta^\circ = 22^\circ 37'.$$

Again here, we need to write the answer as a decimal number. We consider the $37'$. In the table $37'$ is very close to $36'$. This means that it could be written as 0.6° .

Therefore, $22^\circ 37'$ is now written as 22.6° .

Hence, the angle $\theta = 22.6^\circ$.

Note that if we use the other procedure, we should come up with the same answer. We are dealing with the same angle, hence it should never change whichever procedure you use.

For the sake of comparison, we shall use both procedures.

Using Procedure B, we take 0.9232. The angle for this ratio (0.9232) is $22^\circ 36'$.

We then have to find the difference between the numbers.

$$\text{Difference} = 0.9232 - 0.9231$$

$$\text{Difference} = 1$$

Note that this difference 1 is not available in the difference column. We take the closest to 1, that is, 3.

This difference 3 is directly below $1'$. This difference is then added to the angle.

$$\text{Then, } \theta^\circ = 22^\circ 36' + 1'$$

$$\text{Therefore, } \theta^\circ = 22^\circ 37'$$

$$\text{Hence, } \theta = 22.6^\circ$$

Using the Scientific Calculator, the following is the process:

$$12 \div 13 = 0.9231 \quad \text{2ndF} \quad \text{Cos} \quad 22.61989..$$

The final answer is rounded off to 3 significant figures. This means that you round off the answer to 1 decimal point. Therefore, the answer becomes: $\theta = 22.6^\circ$.

We have come to the end of this part in finding the angles of given trigonometry ratios. We have seen how to calculate the angles given any ratio.

In the next part, we will now consider finding the ratios of the given angles under trigonometry.



The cosine ratios decrease from 0° to 89.9° . You will notice that the ratios decrease from 1.000 at 0° going down to 0.0017 at 89.9°

Hence, we add if the ratio picked is before **our ratio (as in Procedure B)** and we subtract if the ratio picked is after **our ratio (as in Procedure A)**.

Finding the Sine, Cosine and the Tangent Ratios of Given Angles

Let us begin by considering the following angle in all the cases. For our discussion purpose we will use angle 40° .

Example 1

Find the ratios of the following:

Sin 40° , Cos 40° and tan 40° .

Solution

We are starting with Sin 40° .

We will write it as:

$$\text{Sin } 40^\circ = x$$

From the angle of Sine Table; we check for 40° under 0° . Would you say why we are saying under 0° ?

We are sure you said that it is because the angle is a whole number with no decimal part.

From the tables;

$$\sin 40^\circ = 0.6428$$

Using the same procedure, we find that:

$$\cos 40^\circ = 0.7660$$

$$\tan 40^\circ = 0.8391$$

You should notice that if you are using a scientific calculator, you need to just key in the value and press the appropriate trigonometry ratio key as shown in the examples above.

Now let us do one other situation. This time the angle given has a decimal part. This will help you work with angles with decimal parts.

Example 2

Find the value of $\sin 37.6^\circ$.

Solution

If you are using the mathematical tables, here is the procedure:

From the tables,

We check for angle 37° from the left side column. Then we check for 0.6° from the top part of the table. We then draw imaginary lines along 37° and down from 0.6° . Where these two lines meet, the ratio is picked as shown below in the table.

Note that you can use a pencil to draw these lines if you are having problems in drawing the imaginary lines. As long as you remember to erase the pencil marks after you finish the work.

Sines of angles

θ											ADD Differences				
	0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1'	2'	3'	4'	5'
0°	0.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0.0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0.0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	0.0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	0.0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	15
5	0.0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	0.1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	0.1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	0.1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	0.1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10°	0.1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	11	14
11	0.1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	0.2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	0.2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	0.2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	0.2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	0.2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	0.2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	0.3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	0.3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20°	0.3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	0.3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	0.3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	0.3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	0.4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	0.4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	0.4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	0.4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	0.4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	0.4848	4863	4879	4893	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30°	0.5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	0.5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	0.5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	0.5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	0.5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	0.5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	0.5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	0.6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	0.6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	0.6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40°	0.6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	0.6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	0.6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	0.6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	0.6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

Table 3: Extract from the Sines of angles showing the 36.7°

From Table 3, therefore, $\text{Sin } 37.6^\circ = 0.6101$

Now, if you are using the Scientific Calculator, the procedure will be done so easily as follows:

Some calculator you begin with the number first then the Ratio.

You key in **37.6** first followed by **Sin** as shown:

$$\boxed{37.6} \quad \boxed{\text{Sin}} \quad = \quad 0.6101$$

Therefore, $\text{Sin } 37.6 = 0.6101$

Other calculators, you begin by keying in the ratio first followed by the number as shown below:

You key in **Sin** first followed by **37.6** as shown:

$$\text{Sin } \boxed{37.6} = 0.6101$$

Let us consider another situation. This time we are to use an angle with minutes expressed in the question. This is to help you understand how to work with minutes given in angles. You should note that this method only works with mathematical tables.

Example 3

Write down the ratio of Cos $22^{\circ} 48'$.

Solution

From the table of cosine of angles, we should check for 22° from the left side column and then for $48'$ from the top as shown below.

Cosines of angles

θ	$0'$	$6'$	$12'$	$18'$	$24'$	$30'$	$36'$	$42'$	$48'$	$54'$	SUBTRACT Differences					
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1'	2'	3'	4'	5'	
0	1.000	1.000	1.000	1.000	1.000	1.000	.9999	.9999	.9999	.9999	0	0	0	0	0	
1	0.9998	0.9998	0.9998	0.9997	0.9997	0.9997	0.9996	0.9996	0.9995	0.9995	0	0	0	0	0	
2	0.9994	0.9993	0.9993	0.9992	0.9991	0.9990	0.9990	0.9989	0.9988	0.9987	0	0	0	1	1	
3	0.9986	0.9985	0.9984	0.9983	0.9982	0.9981	0.9980	0.9979	0.9978	0.9977	0	0	1	1	1	
4	0.9976	0.9974	0.9973	0.9972	0.9971	0.9969	0.9968	0.9966	0.9965	0.9963	0	0	1	1	1	
5	0.9962	0.9960	0.9959	0.9957	0.9956	0.9954	0.9952	0.9951	0.9949	0.9947	0	1	1	1	2	
6	0.9945	0.9943	0.9942	0.9940	0.9938	0.9936	0.9934	0.9932	0.9930	0.9928	0	1	1	1	2	
7	0.9925	0.9923	0.9921	0.9919	0.9917	0.9914	0.9912	0.9910	0.9907	0.9905	0	1	1	2	2	
8	0.9903	0.9900	0.9898	0.9895	0.9893	0.9890	0.9888	0.9885	0.9882	0.9880	0	1	1	2	2	
9	0.9877	0.9874	0.9871	0.9869	0.9866	0.9863	0.9860	0.9857	0.9854	0.9851	0	1	1	2	2	
10	0.9848	0.9845	0.9842	0.9839	0.9836	0.9833	0.9829	0.9826	0.9823	0.9820	1	1	2	2	3	
11	0.9816	0.9813	0.9810	0.9806	0.9803	0.9799	0.9796	0.9792	0.9789	0.9785	1	1	2	2	3	
12	0.9781	0.9778	0.9774	0.9770	0.9767	0.9763	0.9759	0.9755	0.9751	0.9748	1	1	2	3	3	
13	0.9744	0.9740	0.9736	0.9732	0.9728	0.9724	0.9720	0.9715	0.9711	0.9707	1	1	2	3	3	
14	0.9703	0.9699	0.9694	0.9690	0.9686	0.9681	0.9677	0.9673	0.9668	0.9664	1	1	2	3	4	
15	0.9659	0.9655	0.9650	0.9646	0.9641	0.9636	0.9632	0.9627	0.9622	0.9617	1	2	2	3	4	
16	0.9613	0.9606	0.9603	0.9598	0.9593	0.9588	0.9583	0.9578	0.9573	0.9568	1	2	2	3	4	
17	0.9565	0.9558	0.9553	0.9548	0.9542	0.9537	0.9532	0.9527	0.9521	0.9516	1	2	3	3	4	
18	0.9511	0.9505	0.9500	0.9494	0.9489	0.9483	0.9478	0.9472	0.9466	0.9461	1	2	3	4	5	
19	0.9455	0.9449	0.9444	0.9438	0.9432	0.9426	0.9421	0.9415	0.9409	0.9403	1	2	3	4	5	
20	0.9397	0.9391	0.9385	0.9379	0.9373	0.9367	0.9361	0.9354	0.9348	0.9342	1	2	3	4	5	
21	0.9336	0.9330	0.9323	0.9317	0.9311	0.9304	0.9298	0.9291	0.9285	0.9278	1	2	3	4	5	
22	0.9272	0.9265	0.9258	0.9252	0.9245	0.9239	0.9232	0.9225	0.9219	0.9212	1	2	3	4	6	
23	0.9203	0.9196	0.9189	0.9181	0.9174	0.9167	0.9160	0.9153	0.9146	0.9139	1	2	3	5	6	
24	0.9135	0.9128	0.9121	0.9114	0.9107	0.9100	0.9092	0.9085	0.9078	0.9070	1	2	4	5	6	
25	0.9068	0.9060	0.9053	0.9046	0.9039	0.9032	0.9025	0.9018	0.9011	0.9003	0.8996	1	3	4	5	6
26	0.8988	0.8980	0.8973	0.8965	0.8957	0.8949	0.8942	0.8934	0.8926	0.8918	1	3	4	5	6	
27	0.8910	0.8902	0.8894	0.8886	0.8878	0.8870	0.8862	0.8854	0.8846	0.8838	1	3	4	5	7	
28	0.8829	0.8821	0.8813	0.8805	0.8796	0.8788	0.8780	0.8771	0.8763	0.8755	1	3	4	6	7	
29	0.8746	0.8738	0.8729	0.8721	0.8712	0.8704	0.8695	0.8686	0.8678	0.8669	1	3	4	6	7	
30	0.8660	0.8652	0.8643	0.8634	0.8625	0.8616	0.8607	0.8599	0.8590	0.8581	1	3	4	6	7	
31	0.8572	0.8563	0.8554	0.8545	0.8536	0.8526	0.8517	0.8508	0.8499	0.8490	2	3	5	6	8	
32	0.8480	0.8471	0.8462	0.8453	0.8443	0.8434	0.8425	0.8415	0.8406	0.8396	2	3	5	6	8	
33	0.8387	0.8377	0.8368	0.8358	0.8348	0.8339	0.8329	0.8320	0.8310	0.8300	2	3	5	6	8	
34	0.8290	0.8281	0.8271	0.8261	0.8251	0.8241	0.8231	0.8221	0.8211	0.8202	2	3	5	7	8	
35	0.8192	0.8181	0.8171	0.8161	0.8151	0.8141	0.8131	0.8121	0.8111	0.8100	2	3	5	7	8	
36	0.8090	0.8080	0.8070	0.8059	0.8049	0.8039	0.8028	0.8018	0.8007	0.7997	2	3	5	7	9	
37	0.7986	0.7976	0.7965	0.7955	0.7944	0.7934	0.7923	0.7912	0.7902	0.7891	2	4	5	7	9	
38	0.7880	0.7869	0.7859	0.7848	0.7837	0.7826	0.7815	0.7804	0.7793	0.7782	2	4	5	7	9	
39	0.7771	0.7760	0.7749	0.7738	0.7727	0.7716	0.7705	0.7694	0.7683	0.7672	2	4	6	7	9	
40	0.7660	0.7649	0.7638	0.7627	0.7615	0.7604	0.7593	0.7581	0.7570	0.7559	2	4	6	8	9	
41	0.7547	0.7536	0.7524	0.7513	0.7501	0.7490	0.7478	0.7466	0.7455	0.7443	2	4	6	8	10	
42	0.7431	0.7420	0.7408	0.7396	0.7385	0.7373	0.7361	0.7349	0.7337	0.7325	2	4	6	8	10	
43	0.7314	0.7302	0.7290	0.7278	0.7266	0.7254	0.7242	0.7230	0.7218	0.7206	2	4	6	8	10	
44	0.7193	0.7181	0.7169	0.7157	0.7145	0.7133	0.7120	0.7108	0.7096	0.7083	2	4	6	8	10	

Table 4: Cosines of angles showing $22^\circ 48'$

From Table 4, therefore, $\text{Cos } 22^\circ 48' = 0.9219$

We can as well work out the following in a similar manner:

- (a) $\text{Sin } 41^\circ 52'$
- (b) $\text{Cos } 41^\circ 52'$

Solution

- (a) $\text{Sin } 41^\circ 52'$

From the tables, we have to check for 41° on the left side column and $52'$ form the top part of the tables.

Sines of angles

θ											ADD Differences				
	0'	1'	2'	3'	4'	5'	6'	7'	8'	9'	1'	2'	3'	4'	5'
0°	0.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0.0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0.0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	0.0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	0.0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	15
5	0.0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	0.1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	0.1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	0.1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	0.1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10°	0.1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	11	14
11	0.1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	0.2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	0.2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	0.2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	0.2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	0.2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	0.2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	0.3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	0.3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20°	0.3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	0.3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	0.3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	0.3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	0.4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	0.4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	0.4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	0.4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	0.4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	0.4848	4863	4879	4893	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30°	0.5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	0.5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	0.5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	0.5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	0.5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	0.5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	0.5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	0.6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	0.6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	0.6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40°	0.6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	0.6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	0.6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	0.6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	0.6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

Table 5: Extract from Sines of angles showing Sin $41^\circ 52'$

From the table 5, we see that $52'$ is not among the numbers we have. But that $52'$ is between $48'$ and $54'$.

We have to pick from the two numbers one which is the nearest to $52'$. In this case, it is $54'$.

Hence, the ratio picked is 0.6678 which is under $54'$. But this number is not for $52'$. So we have to find the difference between $54'$ and $52'$.

Difference = $54' - 52'$

Difference = $2'$

When we check in the **top row** of the last column for 2' and then mark a line down to meet the line from 41° , we then pick the number at the intersection of the two lines, which is 4.

Since the number picked is ahead of our number, we subtract the 4 from the ratio picked.

$$\sin 41^\circ 52' = 0.6678$$

$$\underline{- 4}$$

$$\mathbf{0.6674}$$

Therefore, $\sin 41^\circ 52' = 0.6674$.

We should note that this ratio, 0.6674 is between 0.6665 and 0.6678, although it is not seen directly from the table.

Note also that if we picked the earlier number which is 0.6665, we should have added the difference as shown below:

$$\text{Difference} = 52' - 48'$$

$$\text{Difference} = 4'$$

Now 4' gives us 9 as shown by the **dotted line**.

$$\sin 41^\circ 52' = 0.6665$$

$$+ \underline{\quad 9}$$

$$\mathbf{0.6674}$$

Therefore, **$\sin 41^\circ 52' = 0.6674$** .

When we use the scientific calculator, we first convert the minutes to the degrees, then we key in the value as shown below:

$$52' = \frac{52}{60} = \frac{13}{15} = 0.867.$$

Since degrees have to be written to one decimal place, 0.867 becomes 0.9.

$$\text{Therefore, } \sin 41^\circ 52' = \sin 41.9^\circ$$

We then key in the number:

$$\sin 41.9^\circ = \mathbf{0.6678}$$

We have got 0.6678 instead of 0.6674 because of rounding off as we worked to convert the 52 minutes to degrees. If we used 0.867 as 41.867, we would have got:

$$\sin 41.867^\circ = 0.6674.$$



The following activity is to help you understand and master the skills of finding the ratio values using the mathematical tables or the use of the scientific calculator.

Activity 3

Write down the following:

(a) $\cos 41^\circ 52'$

(b) $\tan 41^\circ 52'$



Here is the feedback for the work above. You should check it through after you have done the activity. The feedback is meant to help you check your work and to identify your problems that you may be encountering. It is therefore very important for you to check with the feedback after you have worked through the work.

Feedback

We are working with minutes. You should remember to check them in the tables

(a) $\cos 41^\circ 52'$

$$\cos 41^\circ 52' = 0.7455$$

$$\underline{0.7447}$$

Therefore, $\cos 41^\circ 52' = 0.7447$



For cosine ratios we subtract when the number picked is earlier (or comes first) and we add when the number picked is later (or comes after our number).

$$(b) \tan 41^\circ 52'$$

$$\tan 41^\circ 52' = 0.8941$$

$$+ \quad \underline{21}$$

$$\underline{0.8962}$$

$$\text{Therefore, } \tan 41^\circ 52' = 0.8962$$

We have come to the end of this topic. We have shown how to calculate the trigonometry ratios and how to find the related angles of each ratio. We have also shown how to use the mathematical table and how to use the scientific calculator. What is important is for you to see that you may use any instrument to calculate the trigonometry ratios and the related angles. It all depends on the instrument that you have available. You should not be restricted to use only one instrument when you can use any.

You can now read the topic summary below.

Topic 1 Summary



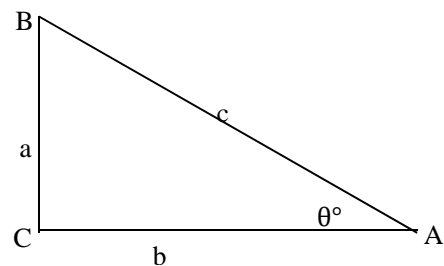
In this topic, you learnt about:

- Trigonometry ratios which are defined as follows:

$$\checkmark \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

$$\checkmark \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$

$$\checkmark \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$$



You went ahead to learn the following issues in the topic:

- How to find the **ratio** values of sine, cosine and tangent using mathematical tables or the scientific calculator.
- How to determine the **angle** size of the trigonometry ratio under sine, cosine and tangent.

As stated above the use of the mathematical tables or the scientific calculator lies in you as an individual. What has been done above is to show you by going through all the steps of the procedures of using, especially the mathematical tables. It is however, recommended that you acquire a **scientific calculator**. It makes work easy as shown in the work.

You were also encouraged to answer all the questions in the activities provided in this topic. The activities are intended to help you assess how well you understood the content of the various parts of the topic. If you did not do very well in the activity, this means that you needed to go over the material again before you attempt the topic exercise.

Now that you are ready, attempt topic exercise 1 in the assignment section. You should then check the feedback to correct your answers

In Topic 2 we will deal with the application of trigonometric ratios.

Topic 2 Application of the Trigonometric Ratios

This is the second and last topic of unit 13. In the first topic, you learnt to define trigonometry ratios. You learnt how to derive or come up with the three trigonometry ratios. You went further to learn how to find the values of the three ratios using the mathematics table or the scientific calculator. The previous topic has been the basis of the topic you are about to begin to study. You will use the knowledge and information you have gained to answer and solve problems in this section of the unit. You will learn how to calculate for a missing length of a side and an unknown angle. These are relatively new to you as you might not have learnt about them in Junior Certificate program. You learnt scale drawing and three figure bearings and used this to solve problems involving Angles of elevation and depression. This time you will learn how to solve problems involving angles of elevation and depression using trigonometry ratios. You are going to learn to apply the ratios in situations that need to be resolved. You should remember that the concept of ratios has already been tackled. Here you will learn how to use any of the three ratios to solve situational problems.

After studying the topic you will be required to answer self-assessment questions at the end of the topic. You are encouraged not to go through the feedback before doing the topic exercises.

It is important for you to understand that the trigonometric ratios are very useful in life. These are used to estimate distances or heights when there is knowledge of a reference point such as a point on the level ground. It may seem strange at first but as you continue to work and practice, you will be familiar with the trigonometry ratios, and then you will see their usefulness in life.

In this second topic of the unit, we will address the last two of the four unit outcomes.



- *Apply* trigonometric ratios to calculate angles, height and distances.
- *Apply* trigonometric ratios to solve problems involving three figure bearing.

You will also achieve the following objectives.



- *Find* the length of a side
- *Find* the value of the unknown angle

Application of the Trigonometry Ratios

Among numerous applications, the three ratios, that is, Sine, Cosine and Tangent, can be practically used to find missing sides or missing angles in given triangle situations.

We will consider two such situations. The first of which is finding the length of a side. From there, we will go on to finding the missing angle in a given triangle.

In the first part of the topic, you learnt how to use the mathematical tables and the scientific calculator. You need to use that knowledge and skill in this part of the topic. You will notice that the illustration of keying in the details in the calculator will not be shown any more. This is because it is expected you to use the knowledge acquired.

Finding the Length of a Side

The question in the example below is to introduce you to the application of the trigonometry ratios. This will help you appreciate the process and be able to use it in any situation.

Example 1

Find the length of the side labelled x in the triangle shown below.

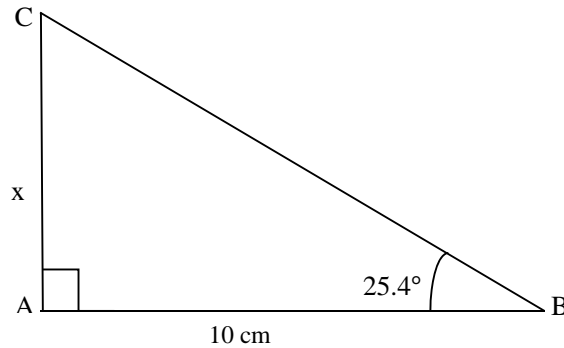


Figure 2.1: Example 1 - Finding the length of a labelled triangle side

Solution

In this example, we know two sides, that is, the opposite side to the angle and the adjacent side to the angle. From this information, we can use one of the ratios. Would you know which one? Write your answer below.

It is assumed you wrote: **Tangent of an angle.**

You are right. The ratio to be used here is the tangent of the angle given in the diagram. Having known all the values, we substitute them in the formula:

$$\text{Tan } \theta^\circ = \frac{\text{Opposite}}{\text{Adjacent side}}$$

$$\text{Tan } 25.4^\circ = \frac{\text{Opposite}}{\text{Adjacent side}}$$

$$\text{Tan } 25.4^\circ = \frac{x}{10}$$

$$x = 10 \times \text{Tan } 25.4^\circ$$

Cross multiply:

$$\text{Tan } 25.4^\circ \times 10$$

From the Tangent Tables in the mathematics tables, you will get $\tan 25.4^\circ = 0.4748$. If we keyed in this value in the calculator, we should be able to get the same ratio: 0.4748. We substitute this value:

$$x = 10 \times 0.4748$$

$$x = 4.748$$

Therefore, $x = 4.75 \text{ cm}$ (3 Sig. figures)

Let us now work out the next question and ensure that you have understood the principle explained in the first example. This will help you appreciate the process and be able to use it in any situation.

Example 2

Find the length of the side labelled **b** in the diagram shown below.

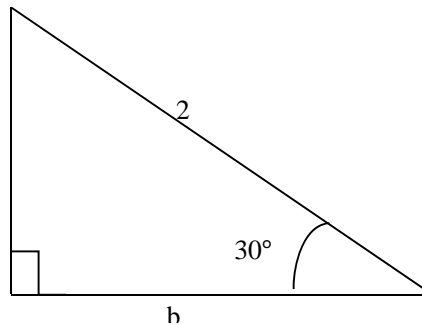


Figure 2.2: Example 2 - finding the length of a labelled triangle side

Solution

From figure 4.2, we are given the adjacent side to the angle and the hypotenuse. This means we are to use the cosine ratio.

$$\cos \theta^\circ = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$$

$$\cos 30^\circ = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$$

$$\cos 30^\circ = \frac{b}{2}$$

Cross multiply: $\cos 30^\circ \times 2$

$$\text{Then, } b = 2 \times \cos 30^\circ$$

From the Cosine Table or the calculator, $\cos 30^\circ = 0.8660$

$$\text{Therefore, } b = 2 \times 0.8660$$

$$b = 1.732$$

Therefore, **$b = 1.73$ units** (3 Sig. figures)

Let us now work out the next question. It is basically the same principle as example above. This is to emphasise the concept and it will help you appreciate the knowledge of trigonometry.

Example 3

Find for the value of y in the diagram.

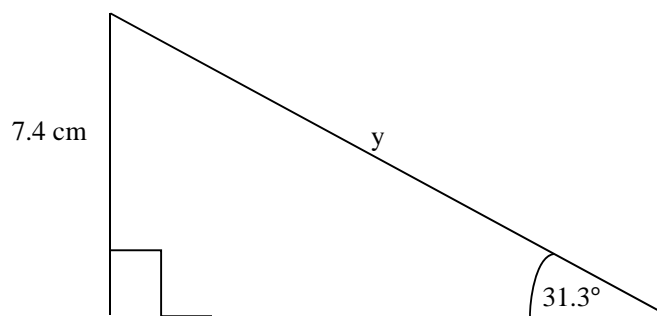


Figure 2.3: Example 3 – finding the value of y

Solution

From figure 4.3, the sides to use are the hypotenuse and the opposite side. This means that we will use the Sine ratio.

$$\sin \theta = \frac{\text{Opposite side}}{\text{Hypotenuse}}$$

$$\sin 31.3^\circ = \frac{\text{Opposite side}}{\text{Hypotenuse}}$$

$$\sin 31.3^\circ = \frac{7.4}{y}$$

$$\sin 31.3^\circ \times y = 7.4$$

$$\frac{\cancel{\sin 31.3^\circ} \times y}{\cancel{\sin 31.3^\circ}} = \frac{7.4}{\sin 31.3^\circ}$$

Divide both sides by $\sin 31.3^\circ$

$$\text{Therefore, } y = \frac{7.4}{\sin 31.3^\circ}$$

We know that $\sin 31.3^\circ = 0.5195$, and substituting it in the equation:

$$\text{Therefore, } y = \frac{7.4}{0.5195}$$

You can use an ordinary calculator to divide 7.4 by 0.5195

$$y = 14.244$$

Therefore, **y = 14.2 cm** (3 Sig. figures)

The next example is on the use of trigonometry ratios in solving problems involving three Figure bearings. It will show you how to apply the ratios in solving such problems.

Example 4

A village is 8 Km on a bearing of 040° from a point O. Calculate how far the village is north of O.

Solution

You should use free hand to sketch a diagram to represent the information given above in the question. First we have to sketch this situation.

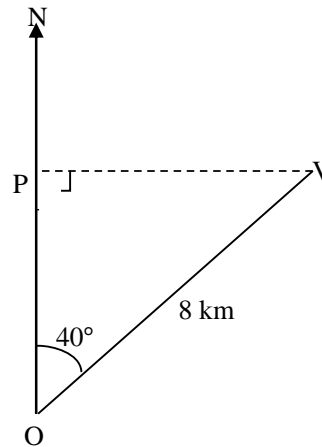


Figure 2.4: Sketch on calculating distance

From the diagram and the information given to us, we need to find the distance OP. OP is adjacent to the known angle; therefore we shall use the cosine ratio.

$$\text{Cos } \theta^\circ = \frac{\text{OP}}{\text{OV}}$$

$$\text{Cos } 40^\circ = \frac{\text{OP}}{\text{OV}}$$

We know that OV is 8 Km.

$$\text{Cos } 40^\circ = \frac{\text{OP}}{8}$$

$$\text{OP} = 8 \times \text{Cos } 40^\circ$$

$$\text{Cos } 40^\circ = 0.7660$$

$$\text{OP} = 8 \times 0.7660$$

$$\text{OP} = 6.128$$

Therefore, the distance **OP = 6.13 Km**(3 Sig. Figures)

From the mathematics table: Cosine table or calculator.



Now that we have done 4 examples, you can now be confident to tackle the following activity. This activity is to help you improve the working skills you have learnt from the 4 examples.

Activity 1

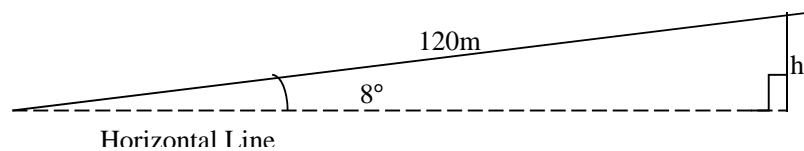
A car travels 120 m along a straight road which is inclined at 8° to the horizontal. Calculate the vertical distance through which the car rises.



Here is the feedback for the work above. You should check it through after you have done the activity.

Feedback

Again we need to sketch the situation given to us.



Here the distance we are to find is opposite to the angle given.
Hence we are to use the sine ratio.

$$\sin 8^\circ = \frac{h}{120}$$

$$\sin 8^\circ = \frac{h}{120} \text{ (we cross multiply)}$$

$$\sin 8^\circ \times 120 = h$$

$\sin 8^\circ = 0.1392$ (from the Sine Table or Scientific Calculator)

$$0.1392 \times 120 = h$$

$$16.704 = h$$

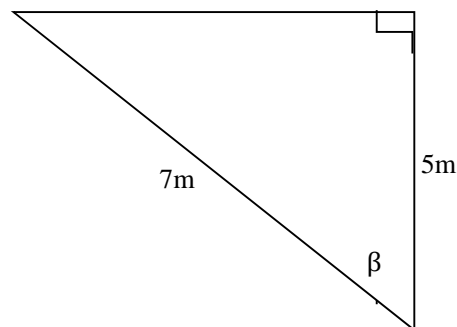
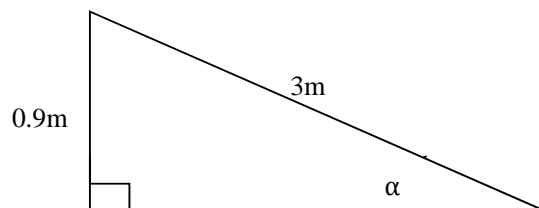
Therefore, the **vertical height = 16.7 m** (3 Sig. Figures)

Finding an Unknown Angle

The following example will help you to work with angles using the Trigonometry ratios. It now introduces you to finding the missing angle in a given situation.

Example 5

Calculate the angles α , β in the figures given below.



Figures 2.5 and 2.6: Calculation of the angles α , β

Solution

In the first diagram, Figure 2.5, we are to use sine ratio.

$$\sin \alpha = \frac{\text{Opposite side}}{\text{Hypotenuse}}$$

$$\sin \alpha = \frac{0.9}{3}$$

$$\sin \alpha = 0.3$$

In using the sine tables or the scientific calculator, this number 0.3 will be considered as 0.3000 as in the tables.

$$\sin \alpha = 0.3000$$

$$\text{Therefore, } \alpha = \sin^{-1} 0.3000$$

$$\text{Therefore, } \alpha = 17^\circ 27'$$

Remember how to change to decimal form. We discussed it in the earlier examples. Use that knowledge to change 27' to decimal number in degrees.

In the second diagram, figure 2.6, we are given the adjacent side and the hypotenuse side. Hence, we are to use the cosine ratio.

$$\cos \beta = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$$

$$\cos \beta = \frac{5}{7}$$

$$\cos \beta = 0.7143$$

$$\beta = \cos^{-1} 0.7143$$

From the Cosine Tables or the Scientific calculator,

$$\beta = 44^\circ 25'$$

$$\text{Therefore, } \beta = 44^\circ 25'.$$

You should note that in all the above examples and the activity, the scientific calculator gives the number to a decimal number without the minutes. You should realise that the decimal part is the minute part though it is given in different from the usual form.

The next activity is on issues that are part of our everyday activities. You will learn how to use and apply trigonometry in the situations.



The next activity is to help to work with the situation problems as shown below in the activity. This activity is on angles of elevation and angles of depression.

Activity 2

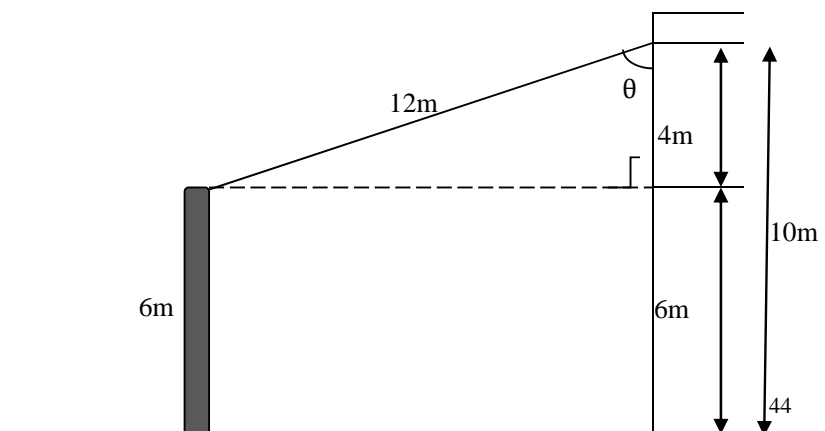
A stretched wire 12 m long goes from the top of a 6-metre pole to a point on the vertical wall 10 m above the ground. Calculate the angle between the wire and the wall?



Here is the feedback for the work above. You should check it through after you have done the activity.

Feedback

We have to sketch this information.





In the diagram, we will use the top triangle. In this work, we are dealing with triangles. The top part of the diagram gives us a right-angled triangle. In this triangle, we are given the adjacent side and the hypotenuse. We are to use the cosine ratio.

$$\cos \theta = \frac{\text{adjacent side}}{\text{Hypotenuse}}$$

$$\cos \theta = \frac{4}{12}$$

$$\cos \theta = 0.3333$$

$$\theta = \cos^{-1} 0.3333$$

$$\theta = 70^{\circ} 32'$$

*Therefore, angle between the wire and the wall is **70°32'***

We have come to the end of of topic 2. We have discussed the calculation of the missing angle. By now, you should be able to convert from minutes to degrees, which will be in the form of a decimal number.

Read through the summary to review what you have learnt so far.

Topic 2 Summary



In this second topic, you learnt how to apply the trigonometry ratios in solving problems. We looked at finding the length of a side of a triangle. You went further to learn how to find the unknown angle in a given triangle.

You were also encouraged to answer all the questions in the activities provided in this topic. The activities are intended to help you assess how well you understood the content of the various subsections of the topic. If you did not do very well in the activities, this means that you needed to study the material again.

Now, do exercise 2 in the assignment section. Again it is a reminder to you not to check the feedback before you attempt the work.

Unit Summary



In topic 1 you learned to define trigonometric ratios. You learnt how to find the values of the ratios and vice versa.

In topic 2, you learned how to apply the trigonometric ratios to find the missing angle or the unknown side. You also learnt how to apply the trigonometric ratios to three figure bearing situations.

By now you would have also completed the topic 2 exercise. This means that, besides the activities that you did within the topics, you have assessed your progress at the end of each of the two topics.

Congratulations on completing unit 13 of mathematics 11. We trust that the activities and topic exercises will adequately help you to assess your own progress. You should complete and submit a tutor marked assignment now, which you will find at the end of the topic exercises.

The next unit, unit 14, is on statistics. You will learn how to collect, classify and present data in the various forms and interpret the data.

References



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4. Rayner D. 1988.*General Mathematics-Revision and Practice*. Oxford: Oxford University Press,

Assignment

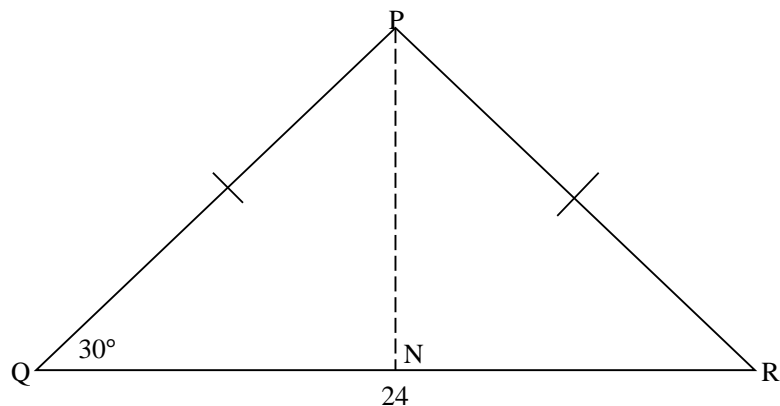


There are two assignments in this unit. These have been expressed as topic exercises. Please note that you should only work on the topic that you are ready to attempt after you have studied the section containing the topic. Do not check through the feedback before you attempt the exercise. This will help you appreciate your level of competence of the content that you have studied so far.

Exercise 1

1. Use mathematical tables to find the value of each of the following:
 - (a) $\sin 14^\circ$
 - (b) $\sin 79^\circ 45'$
 - (c) $\cos 62.5^\circ$
 - (d) $\cos 58^\circ 19'$
 - (e) $\tan 37.5^\circ$
 - (f) $\tan 84^\circ 46'$

2. PQR is an Isosceles triangle in which $PQ = QR$, angle $Q = 30^\circ$ and $QR = 24$ cm. Calculate the length of the perpendicular from P to QR meeting at N.

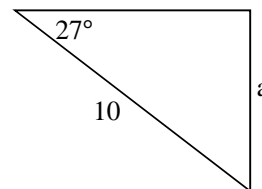


3. A farmer is placing a bee hive up a vertical tree. He is able to see a rabbit at angle of depression of 60° . If the height of the tree is 9 metres, calculate the shortest distance between the farmer and the rabbit.
4. Towela would like to draw a rectangle in which the diagonals will be 18 cm long and which will intersect at 60° . What will be the length and breadth of the rectangle?

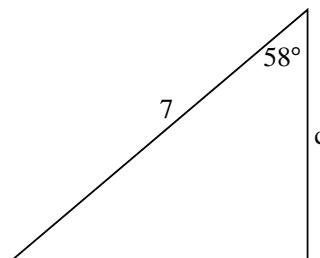
Exercise 2

- 1 Find the length of the sides marked with a letter. Give your answer to three significant figures. Units are cm.

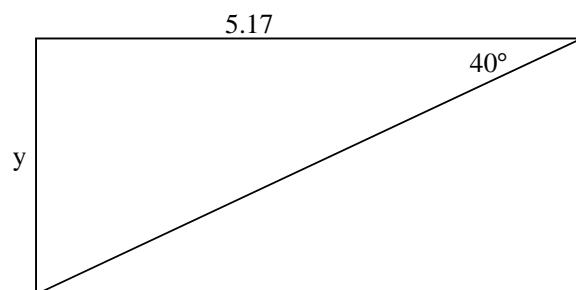
(a)



(b)



(c)



- 2 In the following questions, the triangles have a right-angle at the middle letter.
- (a) In triangle ABC, $\hat{C} = 40^\circ$, BC = 4 cm. Find AB
- (b) In triangle DEF, $\hat{F} = 35.3^\circ$, DF = 7 cm. Find ED
- (c) In triangle PQR, $\hat{R} = 70^\circ$, PR = 12 cm. Find QR.
- (d) In triangle JKL, $\hat{L} = 55^\circ$, KL = 8.21 cm. Find JK
- (e) In triangle KLM, $\hat{K} = 72^\circ 51'$, KL = 5.04 cm. Find LM
- 3 A village is 10 km on a bearing of 050° from a point O. Calculate how far the village is north of O.
- 4 A tightly stretched wire goes from a point on horizontal ground to the top of a vertical pole. If the wire is 8 m long and is inclined at 68° to the horizontal, calculate the height of the pole.
- 5 A stone rolls 300m down a slope. As it falls, it drops 120 m vertically. Calculate the angle of the slope.
- 6 A stone is suspended from a point P by a piece of string 50 cm long. It swings back and forward. Calculate the angle the string makes with the vertical when the stone is 35 cm vertically below P.

Answers to Exercises

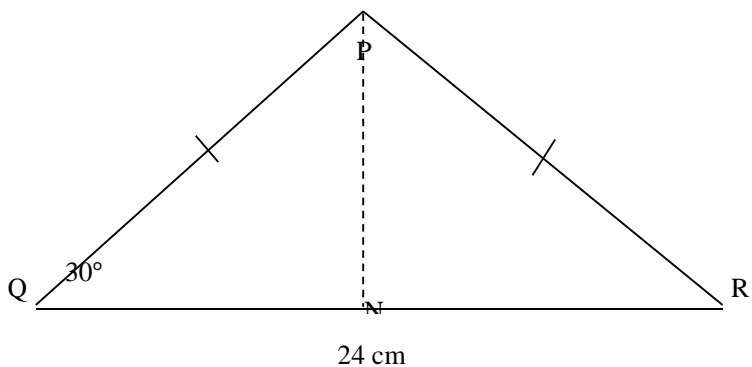


Feedback

When you have done the work above, you should compare your answers with these below. The work may be very different from your work but the final answer should be the same. This means that in mathematics, there are several ways in which a problem can be solved and not all can be written down. Feel free to use any other method you may have come across.

Exercise 1

1. (a) $\sin 14^\circ = 0.2419$
- (b) $\sin 79^\circ 45' = 0.9841$
- (c) $\cos 62.5^\circ = 0.4617$
- (d) $\cos 58^\circ 19' = 0.5253$
- (e) $\tan 37.5^\circ = 0.7673$
- (f) $\tan 84^\circ 46' = 10.99$ (notice that the difference is ignored)



2. Let the point of meeting between QR and the Perpendicular bisector be N

Therefore, $QN = 12$ cm (half of QR)

We know that $\tan 30^\circ = \frac{\text{opp.}}{\text{Adj.}}$

$$\tan 30^\circ = \frac{PN}{QN}$$

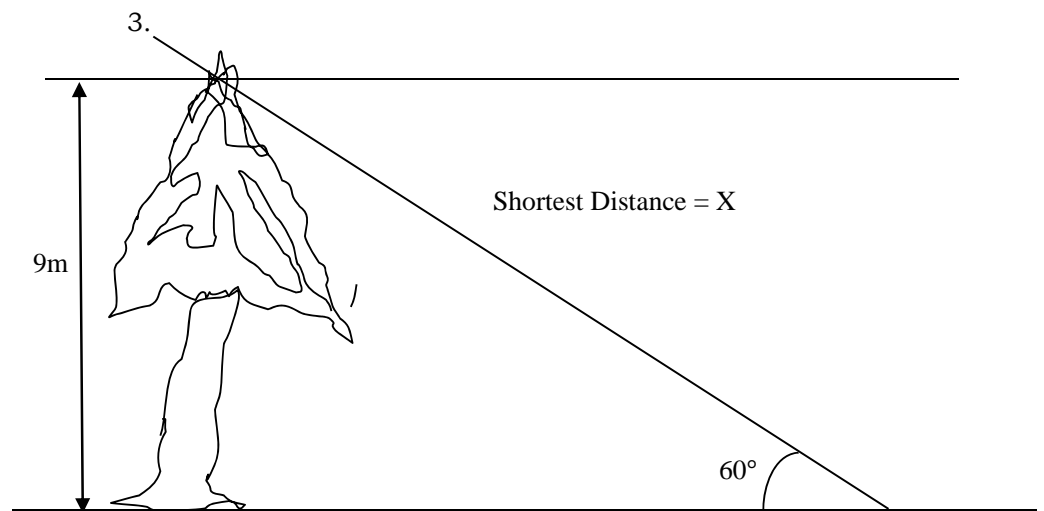
$$\tan 30^\circ = \frac{PN}{12}$$

$$12 \times \tan 30^\circ = PN$$

We know that $\tan 30^\circ = \frac{1}{\sqrt{3}}$

$$\text{Therefore, } PN = 12 \times \frac{1}{\sqrt{3}}$$

$$\mathbf{PN = \frac{12}{\sqrt{3}}}$$



We notice that the shortest distance is the hypotenuse side and we are given the opposite side.

$$\sin 60^\circ = \frac{\text{Opp.}}{\text{Hyp.}}$$

$$\sin 60^\circ = \frac{9}{x}$$

$$\sin 60^\circ \times x = 9$$

$$\mathbf{X = \frac{9}{\sin 60^\circ}}$$

We know $\sin 60^\circ = \frac{\sqrt{3}}{2}$

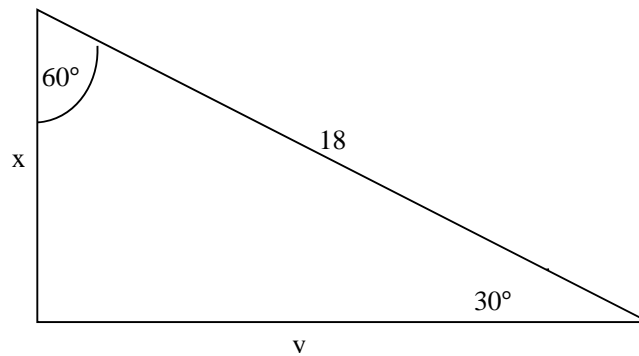
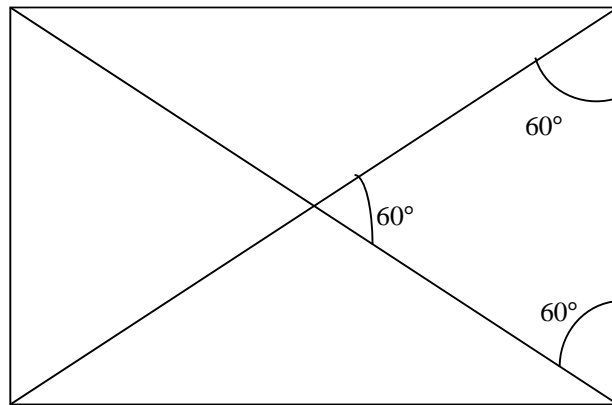
$$X = \frac{9}{\frac{\sqrt{3}}{2}}$$

$$X = 9 \times \frac{2}{\sqrt{3}}$$

$$X = \frac{18}{\sqrt{3}}$$

The shortest distance is $\frac{18}{\sqrt{3}}$ metres.

4. Sketching the information.



$$\sin 30^\circ = \frac{\text{Opp.}}{\text{Hyp.}}$$

$$\sin 30^\circ = \frac{x}{18}$$

$$X = \sin 30^\circ \times 18$$

we know that $\sin 30^\circ = \frac{1}{2}$

$$X = 18 \times \frac{1}{2}$$

$$X = 9$$

Therefore, **the breadth is 9 metres.**

$$\sin 60^\circ = \frac{\text{Opp.}}{\text{Hyp.}}$$

$$\sin 60^\circ = \frac{y}{18}$$

$$Y = \sin 60^\circ \times 18$$

$$\text{We know that } \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$Y = 18 \times \frac{\sqrt{3}}{2}$$

$$Y = 9\sqrt{3}$$

Therefore, **the length is $9\sqrt{3}$**

Exercise 2

$$1. (a) \quad \sin 27^\circ = \frac{\text{opp.}}{\text{Hyp.}}$$

$$\sin 27^\circ = \frac{a}{10}$$

$$\sin 27^\circ \times 10 = a$$

$$0.4540 \times 10 = a$$

$$4.54 = a.$$

Therefore, **a = 4.54**(to 3 Sig, figures)

$$(b) \quad \cos 58^\circ = \frac{\text{Adj.}}{\text{Hyp.}}$$

$$\cos 58^\circ = \frac{c}{7}$$

$$\cos 58^\circ \times 7 = c$$

$$0.5299 \times 7 = c$$

$$3.7093 = c$$

Therefore, **c = 3.71**(to 3 Sig. Figures)

$$(c) \tan 40^\circ = \frac{\text{Opp.}}{\text{Adj.}}$$

$$\tan 40^\circ = \frac{y}{5.17}$$

$$\tan 40^\circ \times 5.17 = y$$

$$0.8391 \times 5.17 = y$$

$$4.338 = y$$

Therefore, **y = 4.34**(to 3 Sig. Figures)

2. In these problems here, you need to use a sketch of a right-angled triangle. Do not worry about how they will look like. What is important to note is that you should be able to get the same answer which ever ratio you use, that is, Sine, Cosine or Tangent.

All answers are to 3 significant figures

$$(a) \tan 40^\circ = \frac{AB}{4}$$

$$AB = \tan 40^\circ \times 4$$

$$AB = 0.8391 \times 4$$

$$\mathbf{AB = 3.36}$$

$$(b) \tan 35.3^\circ = \frac{ED}{7}$$

$$ED = \tan 35.3^\circ \times 7$$

$$ED = 0.7080 \times 7$$

$$ED = 4.956$$

$$\mathbf{ED = 4.96}$$

$$(c) \cos 70^\circ = \frac{QR}{12}$$

$$\begin{aligned} QR &= \cos 70^\circ \times 12 \\ QR &= 0.3420 \times 12 \\ QR &= 4.104 \\ \mathbf{QR} &= \mathbf{4.10} \end{aligned}$$

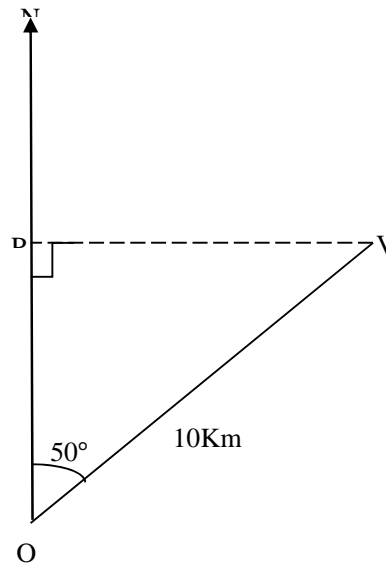
$$(d) \tan 55^\circ = \frac{JK}{8.21}$$

$$\begin{aligned} JK &= \tan 55^\circ \times 8.21 \\ JK &= 1.4281 \times 8.21 \\ JK &= 11.724 \\ \mathbf{JK} &= \mathbf{11.7} \end{aligned}$$

$$(e) \sin 72^\circ = \frac{LM}{5.04}$$

$$\begin{aligned} LM &= \sin 72^\circ \times 5.04 \\ LM &= 0.9511 \times 5.04 \\ LM &= 4.794 \\ \mathbf{LM} &= \mathbf{4.79} \end{aligned}$$

3.



$$\cos 50^\circ = \frac{OP}{10}$$

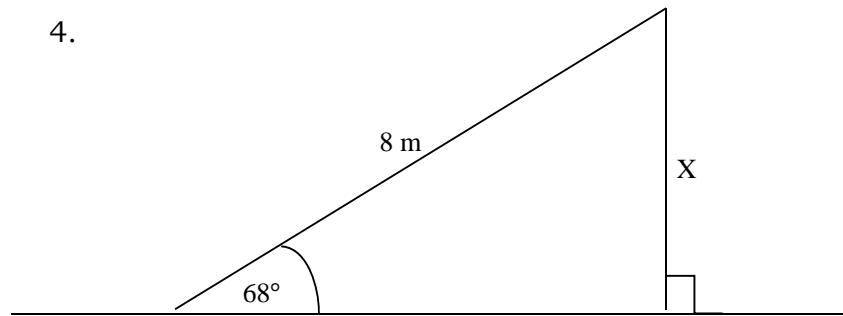
$$OP = \cos 50^\circ \times 10$$

$$OP = 0.6428 \times 10$$

$$OP = 6.428$$

$$\mathbf{OP = 6.43(\text{to 3 Sig. Figures})}$$

4.



$$\sin 68^\circ = \frac{x}{8}$$

$$\sin 68^\circ \times 8 = x$$

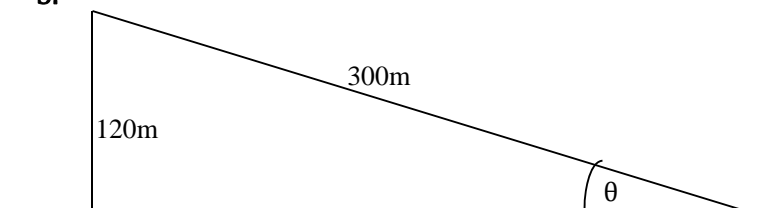
$$0.9272 \times 8 = x$$

$$7.418 = x$$

$$7.42 = x$$

Therefore, $x = 7.42 \text{ m}$

5.



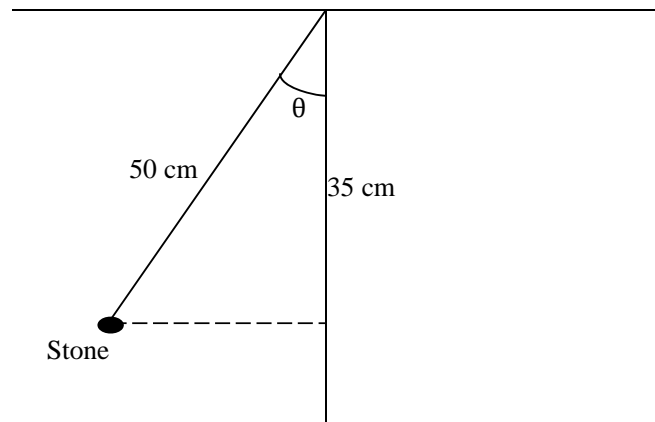
$$\sin \theta = \frac{120}{300}$$

$$\sin \theta = 0.4000$$

$$\theta = \sin^{-1} 0.4000$$

$$\theta = 23^\circ 35' \text{ or } \theta = 23.6^\circ$$

6.



$$\cos \theta = \frac{35}{50}$$

$$\cos \theta = 0.7000$$

$$\theta = \cos^{-1} 0.7000$$

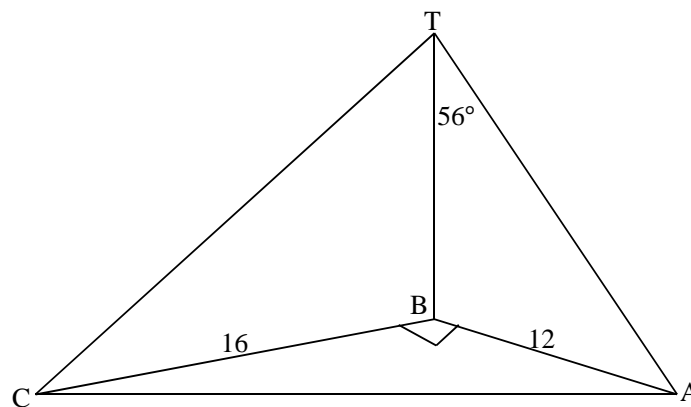
$$\theta = 45^\circ 38' \text{ or } \theta = 45.6^\circ$$

Assessment



This assessment is to help you assess your progress in the unit. You have learnt so much so far but how much of your learning have you understood is the question to be answered in this assessment. The assessment is to help you apply what you have learnt so far in solving situational problems. Remember that this is not a TMA. It is intended as a self-assessment for you. The answers are provided immediately after the questions. However, be honest with yourself and ensure that you do not check the answers before you attempt the assessment.

- In the figure below, A, B, and C are on horizontal ground as shown in the diagram above. T is vertical above B. If $\hat{A}TB = 56^\circ$, $BC = 16$ cm and $AB = 12$ cm, calculate
 - BT
 - TC
 - \hat{BTC}



Source: Mathematics Grade 11 Pupils Book

- An aircraft flies 400 Km from a point O on a bearing of 025° to A and then 700 Km on a bearing of 080° to arrive at B
 - How Far North of O is B?
 - How Far East of O is B?
 - Find the distance and bearing of B from O

Source: General mathematics-Revision and Practice.

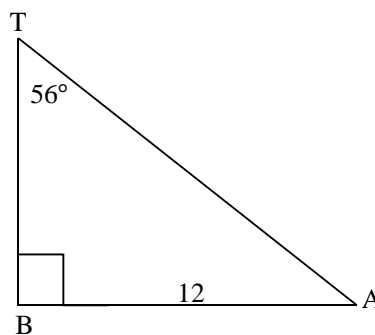


Answers to the self
assessment Activity

When you have done the work above, you should compare your answers with these below. The working may be very different from your working but the final answer should be the same. This means that in mathematics, there are several ways in which a problem can be solved and not all can be written down. Feel free to use any other method you may have come across.

In this question, you should have broken down the diagram into smaller triangles. This enables you to answer this question with ease. We will break down this diagram into three triangles.

1. (a)



Looking at the unknown side, we see that the sides to use are the adjacent side to the angle and the opposite side to the angle. Therefore, we are to use the tangent ratio:

$$\tan \theta = \frac{\text{Opp. side}}{\text{Adj side}}$$

$$\tan \theta = \frac{BA}{BT}$$

$$\tan 56^\circ = \frac{12}{BT}$$

$$\tan 56^\circ \times BT = 12$$

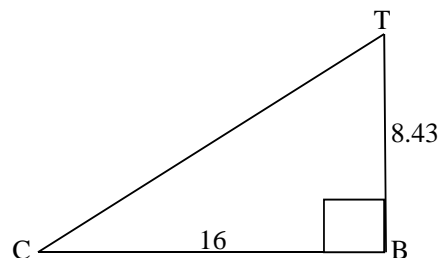
$$1.423 \times BT = 12$$

$$\frac{1.423 \times BT}{1.423} = \frac{12}{1.423}$$

$$BT = 8.433$$

Therefore, the length of BT is **8.43cm** (3 Sig. figures)

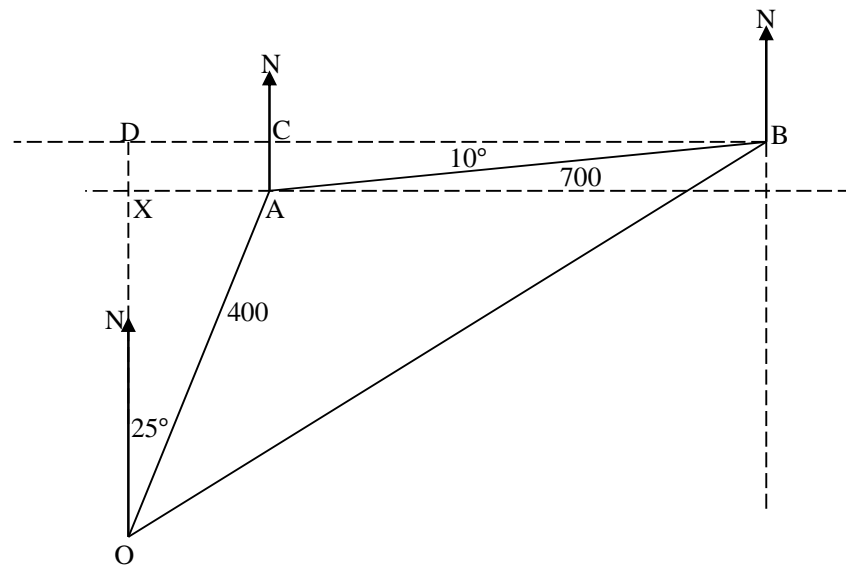
(b)



We use the pythagoras theorem

$$TC^2 = BC^2 + BT^2$$

We substitute the values in the equation. For BT we use the value before rounding off. This helps to maintain accuracy of the final



We will use this diagram throughout this question. You should check frequently as you study through the working to arrive at the answers.

The distance OB north of O is the distance OD. To find the distance of OB north of O (which is the distance OD), we have to use triangle OAX and triangle ACB.

From triangle OAX,

$$\cos 25^\circ = \frac{OX}{OA}$$

$$\cos 25^\circ = \frac{OX}{400}$$

$$\cos 25^\circ \times 400 = OX$$

$$0.9063 \times 400 = OX$$

$$OX = 362.52$$

From triangle ACB

$$\sin 10^\circ = \frac{AC}{AB}$$

$$\sin 10^\circ = \frac{AC}{700}$$

$$\sin 10^\circ \times 700 = AC$$

$$0.1736 \times 700 = AC$$

$$AC = 121.52$$

Now, we know that from the diagram, $AC = XD$. Therefore, $XD = 121.52$ and that OB north of $O = OD$

Hence, $OD = OX + XD$

$$OD = 362.52 + 121.52 = 484.04$$

$$OD = 484$$

Therefore, the distance of B north of O is **484** km.

(b) How far East of O is B.

The distance OB East of O is the distance DB. From your lesson on parallel lines, you should have learnt that the distance between two parallel lines is the same at any two points perpendicularly. In our case, it means that the distance DB is equal to the distance OB east of O. To find the distance of DB East of O (which is the distance DB), we have to use the same triangle OAX and triangle ACB.

From triangle OAX,

$$\sin 25^\circ = \frac{AX}{OA}$$

$$\sin 25^\circ = \frac{AX}{400}$$

$$\sin 25^\circ \times 400 = AX$$

$$0.4226 \times 400 = AX$$

$$AX = 169.04$$

From triangle ACB

$$\cos 10^\circ = \frac{BC}{AB}$$

$$\cos 10^\circ = \frac{BC}{700}$$

$$\cos 10^\circ \times 700 = BC$$

$$0.9848 \times 700 = BC$$

$$BC = 689.36$$

Now, we know that from the diagram, $AX = CD$. Therefore, $CD = 169.04$ and that OB East of $O = DB$

Hence, $DB = DC + CB$

$$DB = 169.04 + 689.36$$

$$DB = 858.4$$

Therefore, the distance of B East of O is **858**km.

(c) To find the distance of OB

We are to use triangle OBD:

We know that $DB = 858$ km and $OD = 484$ km. We are to use the Pythagoras theorem.

$$OB^2 = OD^2 + DB^2$$

$$OB = \sqrt{OD^2 + DB^2}$$

$$OB = \sqrt{484^2 + 858^2}$$

$$OB = \sqrt{234256 + 736164}$$

$$OB = \sqrt{970420}$$

$$OB = 985.1$$

Therefore, the distance from O to B is **985**km.

To find the bearing of B from O, we use the same triangle. We may use any of the ratios:

$$\sin \widehat{NOB} = \frac{DB}{OB}$$

$$\sin \widehat{NOB} = \frac{858.4}{985.1}$$

$$\sin \widehat{NOB} = 0.8714$$

$$\widehat{NOB} = \sin^{-1} 0.8714$$

$$\widehat{NOB} = 60.6^\circ \text{ or } \widehat{NOB} = 60^\circ 36'$$

$$\widehat{NOB} = 61^\circ$$

Therefore, the bearing of B from O is **061** $^\circ$.

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Unit 15

Sequences

Introduction

Welcome to the last unit in your mathematics 11 course. Congratulations on reaching this milestone of your studies! Since this is the last unit, you can now go through the course outcomes again in the course overview section at the beginning of this course. This will help you know how you are progressing in your studies. We are sure that if you worked hard in all the units and assignments, you have gained adequate knowledge and skills to help you solve problems that relate to real life situation. You will also use the knowledge and skills that you have gained to study your in grade 12 mathematics.

In the previous unit we discussed how to present data in the form of bar graph, pie chart, histogram, line graph, and frequency table. You also learned to find mean, mode and median.

In this unit we will discuss sequences. The discussion will help you to address the overall course outcome which requires you to “recognise patterns and structures in a variety of situational forms and justify generalisations”.

Sequences are not new to you in the sense that you learned about them in your Junior Certificate Program. You learned how to determine the rule of the sequences and use it to find the next two numbers in the sequences. The types of sequences you learned included rectangle numbers, triangular numbers, square numbers, and natural numbers.

Sequences are important and interesting topic to learn because they are widely used in our daily lives. One example is the score in a basketball game. The teams score increases by 2. The only fault with this example is that you can score a 3 pointer or foul shot worth 1 point.

(Source:[http://wiki.answers.com/Q/How are arithmetic sequences used in everyday life#ixzz1DN3xiXip](http://wiki.answers.com/Q/How_are_arithmetic_sequences_used_in_everyday_life#ixzz1DN3xiXip))

The unit has two topics. In topic 1 we revise the meaning of sequences in the mathematical context and then recognise a sequence and find the rule that forms the pattern of the sequence.

The last topic is about generalising sequences into simple algebraic statements. We will write the formula that would be used to find the value of any given term in the sequence.

Unit 15 has an end of unit assignment. You will be required to complete the tutor marked assessment 5 based on the last three units and then send it to your tutor for marking.

After doing the tutor marked assignment 5, you are encouraged to do the end of grade 11 course test that comes at the end of this unit.

Upon completion of this unit you will be able to:



Outcomes

- *Recognise and continue* a sequence.
- *Identify* patterns within and across different sequences.
- *Generalise* sequences into simple algebraic statements (including expressions to the n th term).

Timeframe



How long?

We estimate that to complete this unit you will need between 10 and 12 hours. This time includes the time you will spend in doing the activities and checking them against the feedback. If you do not finish studying the unit within this estimated time do not worry since we do not all learn at the same pace.

You are encouraged to spend about 2 hours answering each end of topic exercise in this unit and 2 hours 30 minutes answering tutor marked assignment 5. Since there are two exercises and one tutor marked assignment in this unit, you will spend a total of 6 hours and 30 minutes on these activities.

The total hours for completing the unit will thus be between $16\frac{1}{2}$ and $18\frac{1}{2}$.

Learning Resources



Tip

In order to study this unit with minimal difficulties you will need the following materials:

- A calculator
- A ruler
- A pencil or a pen
- Writing pad

Teaching and Learning Approaches



Tip

In this unit we have used three teaching and learning methods in presenting the content. These methods are:

- **Conceptual:** We will use this method to introduce you to new terms and help you to understand their meanings. We will also guide you to discover the rules and facts forming the sequences and the formulas to find the value of n th terms.
- **Problem-solving:** We will use this method to give you the chance to use the knowledge that you would have gained to solve mathematical problems that relate to real life situation. You will also be encouraged to discuss mathematical problems, answers and strategies with friends.
- **Skills:** Under this method, you will be given some exercises to do which will help you to practice the skills of applying the facts, rules, formulas and procedures in different situations. You will be given self-marked exercises at the end of each topic to do and a Tutor-marked assignment at the end of this unit.

As you read through the unit, do the activities and/or exercises and discuss your ideas with other learners and your tutor. In this way you will be putting into practice these three teaching and learning methods. When you consistently apply these methods, we trust that you will achieve greater understanding of the unit and be able to relate the knowledge to your real life situation.



Terminology

Arithmetic sequence:	A sequence of numbers in which the difference between any two consecutive terms is the same.
Common difference:	The difference between any two successive terms in an arithmetic sequence.
Common ratio:	The constant factor used to multiply consecutive terms in a geometric sequence.
Function:	Variable quantity whose value depends upon the varying values of other quantities.
Geometric sequence:	A sequence in which the consecutive terms are formed by multiplying by a constant factor.
n th:	A last or latest term in a long and tedious series or

	sequence of similar occurrences.
Progression:	Change from term to term according to some law.
Sequence:	A set of numbers arranged in order one after the other according to some rule.
Term:	An element, or number, in the sequence.

(Source: John A. Fyfield and Dudley Blane (2002). *The School Mathematics Dictionary*)

Topic 1: Sequences

In this first topic of the unit, you will learn to define a sequence and find the rule that forms the pattern of the sequence. You will further learn to extend the sequence using the determined rule.

Like in all other topics, after studying through this topic you will be required to work out topic exercise 1 and mark your own work using the feedback provided at the end of the unit. You are encouraged not to go through the feedback before doing the topic exercise.

In this first topic of the unit, we will address the first two of the three unit outcomes, namely: define and continue a sequence, and identify patterns within and across different sequences. We have divided this outcome into the following objectives:



Objectives

- *Define* a sequence.
- *Determine* the rule forming the sequence.
- *Continue* a sequence.
- *Find* a missing term in a sequence.
- *Find* the common difference of the arithmetic sequence.
- *Find* the common ratio of the geometric sequence.

Defining Sequence

As we have already said in the unit introduction, sequences are not new to you because you learned about them in your Junior Certificate Programme. However, you need to revise the meaning of sequence. We encourage you to check the meaning of **sequence** in the English

dictionary and compare it with the one we have given in the terminology section.

For the purposes of our study, we could define a sequence as a series of numbers which are organised in an ordered list. For further understanding let us do the following activity.



Activity 1

Consider a cyclist who cycles at a constant speed of 20 km/h. Using this speed complete the following table:

Time (h)	1	2	3	4	5	6
Distance (km)	20					

Feedback



Feedback

If you have completed the table then that is good. At a constant speed of 20km/h, the cyclist will cover 20 km in 1 hour, 40 km in 2 hours, 60 km in 3 hours, 80 km in 4 hours, 100 km in 5 hours and 120 km in 6 hours.

We can form a list of the distances covered after every hour as shown below:

20, 40, 60, 80, 100...

As we have already defined, the above list of numbers is a sequence.

Now that we have defined the sequence, we can move on to look at the numbers forming the sequence.

What do we call the numbers in a sequence?

The numbers in the sequence are called **terms**. In mathematics, the terms of a sequence are **numbers** or **geometric figures**. You will probably remember that geometric figures are shapes that include points, polygons, and circles and so on. However in this unit we will only deal with numbers. Such a sequence could also be known as a '**numbers sequence**'.

The lists of the whole numbers, natural numbers, even numbers, prime numbers, square numbers, triangular numbers and odd numbers you learned in your Junior Certificate programme are examples of sequences.

You have come to the end of the first part i.e. the definition of the sequence and are now moving on to discuss the next section which will be finite and infinite sequences.

Finite and Infinite Sequences

A sequence may be finite or infinite. Do you remember finite and infinite sets you learned in grade 10? If yes then use the same knowledge you learned on finite and infinite sets to do the following activity in which you will be required to define the finite and infinite sequences.



Activity 2

What do you understand by finite and infinite sequences?

Write down your answer in the space below.

Feedback



Feedback

If you have written down your answer in the space provided, then that is good. As it is given in the terminology section above, a **finite sequence** is a sequence that has the last term. One example of the finite sequence is as follows:

2, 8, 18, 32 ... 98

The last term in this sequence is 98.

The three dots show that the numbers continue and end at 98.

An **infinite sequence** is a sequence that continues without end. One example of infinite sequence is the natural numbers:

1, 2, 3 ...

The three dots at the end show that the numbers continue.

As you go through topic 1, you will notice that only infinite sequences have been discussed. We will deal with finite sequences in topic 2.

As we earlier said in the introduction the numbers in the sequence follow a certain order. Let us now look at how to determine the rule for a sequence.

Determining a Rule for a Sequence

We earlier defined the sequence as a series of numbers which are organised in an ordered list. This means that the numbers forming a sequence form a pattern. Therefore, if we study the way the numbers are arranged, we will be able to write a rule for the relationship between the terms of a sequence.

To understand how to determine a rule for a sequence better, we will do the following activity.



Activity 3

Consider the number sequence 0, 2, 4, 6, 8... Study the relationship between the terms and then find the rule for the sequence.

Write your answer in the space provided below.

Feedback

Feedback

What rule is emerging here?

*If you have written the answer correctly in the space above, then that is good. When we study the number sequence (0, 2, 4, 6, 8...) carefully, we discover that the numbers are even numbers. Therefore, the emerging rule for the sequences is to **add two** to one term in the sequences to find the next term. Thus:*

$$0 + 2 = 2;$$

$$2 + 2 = 4;$$

$$4 + 2 = 6;$$

$$6 + 2 = 8$$

We have added 2 to 0, 2, 4 and 6 to get the sum 2, 4, 6, and 8, respectively.

We have determined a rule for a sequence. We are now ready for the next stage of our study of sequences, namely continuing a sequence.

How do we continue a sequence?

Let us look at the sequence we have just discussed above. The first thing we identified about the sequence is that the sequence is the list of even numbers. The rule for the sequence we found for the pattern of the number is that **two is added** to one term in the sequence to find the next term. We can therefore apply the rule to find additional numbers in the sequence. Therefore, the next two numbers in the sequence can be worked out as follows:

$$8 + 2 = 10$$

$$10 + 2 = 12$$

Therefore, the next two terms are **10 and 12**.

So far we have discussed the meaning of sequence. We have also discussed that if we identify the pattern forming the sequence we can extend it. Now let us look at some of the addition sequences.

Addition Sequence

The addition sequences are sequences in which the same number is added to each term in the sequence to get the next term. To illustrate this let us do the following activity.



Activity 4

Consider the sequence 5, 10, 15, 20, 25 ... Identify the rule for the sequence and use it to find the next number in the sequence.

Write down your working and answer in the space below.

Feedback



Feedback

The numbers in the sequence are increasing by adding 5 to each term of the sequence to get the next term.

First term is 5

Second term is $5 + 5 = 10$

Third term is $10 + 5 = 15$

Fourth term is $15 + 5 = 20$

Fifth term is $20 + 5 = 25$, and so on.

*Adding 5 to 25, we find that the next term in the sequence is **30**.*

The additional sequence in which same number is added to the terms to get the next terms is known as **arithmetic sequence**.

If you check the terminology section above you will find that we have defined arithmetic sequence as a sequence in which the difference of any two consecutive terms is constant or remains the same. This difference is called **common difference**.

How to Find Common Difference

To understand its meaning better, let us look at the following example.

Let us consider the sequence 1, 3, 5, 7..., and find its common difference. We will use the following steps to do this:

Step 1: Let us find the difference between the first two consecutive terms.

$$3 - 1 = 2$$

Step 2: Let us find the difference between the next two consecutive terms and see if we will still get the same answer.

$$5 - 3 = 2$$

In the sequence above, we have discovered that the difference between any two consecutive terms is 2. Let us now look at the sequence in which the terms decrease by the same amount.

Subtraction Sequence

In subtraction sequence, the numbers in the sequence count down by the same number. The sequence begins with a big number. The first step we need to do is work out the number that is being subtracted from the first term to get the next term. We will therefore, use this difference to continue the sequence by subtracting it from the last term.

Now you should do this activity that will help you understand how to continue the sequence that counts down. The method of extending the sequence is similar to the sequence in which numbers increase.



Activity 5

Find the next term in the sequence 56, 49, 42...

Write down your working and answer in the space below.



Feedback

Feedback

This sequence counts down. Let us determine how much the sequence count down each time by subtracting the second term from the first one.

$$49 - 56 = -7$$

We can also try to subtract the next two consecutive numbers to see if we will still get the same answer.

$$42 - 49 = -7$$

Therefore, the rule of this sequence is “count down seven.”

Counting down seven from 42 gives us 35.

$$42 - 7 = 35.$$

We have so far defined the arithmetic sequence and common difference. We further discussed how to calculate the common difference of the sequence. Let us move to the next type of sequence.

Multiplication Sequence

In multiplication sequence, each term in the sequence is multiplied by the same number to get the next term. To illustrate this, let us look at the following activity:



Activity 6

The following sequence is a multiplication sequence. Find the next number in the sequence 5, 10, 20, 40, 80...

Write down your answer in the space below.



Feedback

Feedback

When we examine the numbers, we find that the numbers are increasing. Each term in the sequence is multiplied by 2 as shown below:

First number is 5

Second number is $5 \times 2 = 10$

Third number is $10 \times 2 = 20$

Fourth number is $20 \times 2 = 40$

Fifth number is $40 \times 2 = 80$

Multiplying 80 by 2, we find that the next term in the sequence is **160** as shown below.

$$80 \times 2 = 160$$

We have seen that in multiplication sequence we multiply the terms by the same number to get the next term. This type of sequence is known as **geometric sequence**.

What is geometric sequence?

If you check the meaning of geometric in the dictionary, you will discover that it means increasing or decreasing very rapidly or fast. (Source: Encarta Dictionaries, 2008).

You can as well check the terminology section above and find out how we have defined geometric sequence. We defined geometric sequence as sequences in which the consecutive terms are formed by multiplying by a constant factor. (Source: John A. Fyfield and Dudley Blane (2002). *The School Mathematics Dictionary*)

The word factor is not new to you since we discussed it in grade 10. We defined a factor as one of two numbers that can be multiplied together to give a particular number.

You know the different types of sequences, namely addition, subtraction, multiplication and/or geometric sequences. Our next task is to discuss common ratio.

Common Ratio

The number that is used to multiply the terms in the sequence to get the next term is called a **common ratio**. Thus, the common ratio for the sequence 5, 10, 20, 40, 80... is 2. You learned about the term ratio in Unit 5 where we defined ratio as a comparison between two different quantities or numbers.

Let us do the following example together. The example will help you to understanding how to find the common ratio of the geometric sequence.

Example: Consider the geometric sequence: 1, 3, 9, 27, 81... Find the common ratio of this sequence.

Solution

Since this is a geometric sequence (that is a sequence in which the consecutive terms are formed by multiplying by a constant factor) we will then divide the consecutive terms as shown below.

$$\frac{\text{second term}}{\text{first term}} = \frac{3}{1} = 3$$

Let us divide the third term by second term of the sequence 1, 3, 9, 27, 81..., to see if we will still get the same answer.

$$\frac{\text{third term}}{\text{second term}} = \frac{9}{3} = 3$$

Therefore, we can conclude that the common ratio of the sequence 1, 3, 9, 27, 81... is 3.

We have just completed discussing how to calculate the common ratio of the geometric sequence. We want to introduce to you a new term which you will come across as you study the sequences. This term is geometric progression.

What is geometric progression?

To know what geometric progression is we have to define what progression is. You can check the meaning of progression in the English

dictionary. Progression means change from term to term according to some law. (Source: Macdonald A. M. (1972). Chambers Twentieth Century Dictionary)

We want you to know that geometric sequence and geometric progression means the same.

Extending the Geometric Progression

If we know the common ratio we can extend the geometric sequence or progression to a given number of terms. First we have to find the common ratio by dividing a pair of consecutive terms as we did in the example above. Remember we earlier discussed how to extend the arithmetic sequence in this topic. To understand how to extend the sequence better, let us do the following activity



Activity 7

Find the common ratio of the geometric progression 4, 8, 16, 32, ____ , and use it to find the next term of the sequence.

Write your working and answer in this space.

Feedback



Feedback

We can work out the common ratio of the geometric progression 4, 8, 16, 32, ___ as follows:

$$\frac{\text{second term}}{\text{first term}} = \frac{8}{4} = 2$$

We have to work out the next pair of consecutive terms to see if we will still get 2

$$\frac{\text{third term}}{\text{second term}} = \frac{16}{8} = 2$$

Since we have obtained the same answer in dividing two pairs of consecutive terms, then we can conclude that the common ratio of the geometric progression 4, 8, 16, 32, ___ is 2.

Now we can use the common ratio we have calculated to find the next term in the sequence. We will multiply last term by 2, thus $32 \times 2 = 64$

So far we have discussed how to find the common ratio and extend the geometric sequence. Now let us look at another type of sequence.

Sequences that Increase by a Large Amount

We earlier said in the introduction that in your Junior Certificate Programme you learned about the whole numbers, natural numbers, even numbers, prime numbers, square numbers, triangular numbers and odd numbers. We said that these sets of numbers are examples of sequences. Now let us look at square numbers and find the rule that is used to develop the square numbers apart from squaring the natural numbers.

Consider the sequence 1, 4, 9, 16... Let us find the rule for the sequence and use it to find the next three numbers in the sequence.

The first step we need to do is to find the difference between the consecutive terms. We carry out the calculations as follows:

$$4 - 1 = 3$$

Second term First term

$$9 - 4 = 5$$

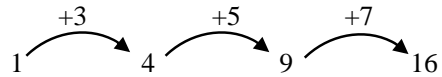
Third term Second term

$$16 - 9 = 7$$

Fourth term Third term

After finding the difference of the consecutive numbers we find 3, 5 and 7. These numbers form the list of odd numbers which increase in size as we move to the right in the sequence.

Diagrammatically, these numbers can be illustrated as shown below.



We can read these numbers as:

In the first arc: $1 + 3 = 4$

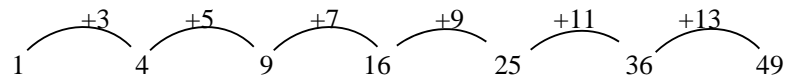
In the second arc: $4 + 5 = 9$

In the third arc: $9 + 7 = 16$

The increase itself forms a sequence we may recognise which is:

3, 5, 7, 9, 11...

We will continue the sequence by adding successively larger odd numbers.



As you can see from the illustration above, the terms in the sequence are increasing by adding the terms of another sequence of numbers that is increasing by 2.

Therefore,

- The first number to use is obtained by adding 2 to 7 to get 9, thus $16 + 9 = 25$;
- The second number to use is obtained by adding 2 to 9 to get 11, thus $25 + 11 = 36$ and
- The third number to use is obtained by adding 2 to 13 to get 15, thus $36 + 15 = 51$.

Therefore, the next three numbers are **25, 36, and 49**.

So far we have discussed that most of the sequences, except for one (the subtraction sequence) were increasing. We discussed the sequences that are additional, multiplication and the sequences that increase by adding the terms of another sequence. Now let us look at the sequence in which the terms are fractions.

Fractional Sequence

Sometimes we have the sequences in which the preceding terms are multiplied by a fraction. Let us consider the following example.

Example

Find the next two terms in the following geometric sequence.

$$1, \frac{1}{2}, \frac{1}{4}, \dots$$

Solution

Since the sequence is the geometric sequence, then, we first have to find the common ratio by dividing the second term by the first term as shown below.

$$\frac{\frac{1}{2}}{1} = \frac{1}{2}$$

We will further divide the third term by second term to see if we will still get the same answer.

$$\frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

We can therefore conclude that the common ratio is $\frac{1}{2}$. Then we will find the next two terms by multiplying the preceding terms by $\frac{1}{2}$ as shown below.

$$\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$

Therefore, the term after $\frac{1}{4}$ is $\frac{1}{8}$.

We will then proceed with our calculations as follows:

$$\frac{1}{8} \times \frac{1}{2} = \frac{1}{16}$$

Therefore, the next two terms of the sequence are $\frac{1}{8}$ and $\frac{1}{16}$.

You have come to the end of topic 1. We hope that you enjoyed working through this topic and you felt motivated to take part in doing the activities in order to strengthen your skills in applying some of the new knowledge which you acquired from our discussions. Now we would like to invite you to read through a topic summary that follows.

Topic 1: Summary



Summary

In order for you to recognise the sequence it was important that you defined what a sequence is and that is why you learned the meaning of the term sequence at the beginning of this topic. You then learned that the elements or numbers in the sequence were known as 'terms'. You also learned how to find the rule for the sequence and used it to extend the sequence. We discussed arithmetic and geometric sequences as well.

Our discussion also included how to find the common difference of arithmetic sequence. The common difference of arithmetic sequence is given as:

Second term = common difference + first term

From our study of this topic, we can use the common difference to extend the sequence. The next term in the sequence can be found by adding the common difference to the preceding term.

We also discussed how to find the common ratio of a geometric sequence. The common ratio of geometric sequence was given as:

$$\frac{\text{second term}}{\text{first term}} = \text{common ratio}$$

We used the common ratio to find other terms in the sequence. The next term in the sequence can be found by multiplying the preceding term by the common ratio.

As part of the learning process you were also encouraged to participate in a number of activities which required you to answer all the questions in each of the activities provided in this topic. The activities are intended to help you assess how well you understood the content of the topic. If you did not do very well in the activity, this means that you need to go over the material again.

In this topic you learned about arithmetic and geometric sequences that are infinite. In the next topic you will learn the sequences that are finite. The last term of the sequences is represented by the letter 'n'. Therefore, in the next topic you will learn to find the nth term of a sequence.

Now you should do topic exercise 1 at the end of the unit to see how much you have learned. After completing the exercise, you should mark your own work by comparing your answers with those provided in the feedback soon after the topic exercise 2. If any of your answers were incorrect, revise the relevant section/s before proceeding to topic 2.

Topic 2: The nth Term of a Sequence

In topic 1 we considered sequences. We discussed the meaning of sequence and the rule that form the pattern of a sequence. We used the rule to extend the sequence. We further discussed how to calculate the common difference of arithmetic sequences and common ratio of the geometric sequences.

In this topic we will discuss the statements in which the letters are used to express the value of the term in the sequence. The letter that is used to represent the unknown term is “n” and is usually called nth term of a sequence.

Upon completion of this second topic of the unit, we will address the last unit outcome which says:



Outcomes

- *Generalise* sequences into simple algebraic statements (including expressions to nth term).

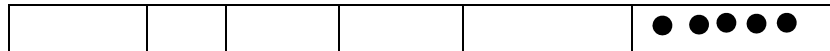
Use of the Letter ‘n’ to Represent the Position of a Term in the Sequence

We can use the letter ‘n’ to represent the position of any term in the sequence in which the rule forming the pattern of the sequence is known. Finding the value of nth term of the sequence is not new to you because you learned it in Junior Certificate Programme. You learned to find the value of nth triangular number.

What are Triangular Numbers?

A triangular number is any number that forms a triangular pattern. Triangular numbers are formed by adding the natural numbers. A list of the first five triangular numbers forming a sequence is 1, 3, 6, 10 and 15. The triangular patterns of these numbers are shown in the table below:

Number	1	3	6	10	15
Pattern					



Look closely at the patterns. You will notice that the values of triangular numbers are obtained by adding the natural numbers as follows:

The value of the first triangular number is 1

The value of the second triangular number is 3 which is obtained by adding 1 and 2.

The value of the third triangular number is 6 which is obtained by adding 1, 2 and 3.

The value of the fourth triangular number is 10 which is obtained by adding the third triangular number and 4. That is 6 and 4.

The value of the fifth triangular number is 15 which is obtained by adding the fourth triangular number and 5. That is 10 and 5.

Instead of adding the natural numbers to find the triangular numbers as we have done above we can use the formula to get the same numbers. The formula we have to use is as follows:

- Value of the n th triangular number = $\frac{n(n+1)}{2}$

Let us use the formula to find the value of the 6th triangular number. In this case, the letter “n” represents 6.

The value of 6th triangular number is $\frac{6(6+1)}{2} = \frac{6 \times 7}{2} = \frac{42}{2} = 21$.

Therefore, the first six triangular numbers are 1, 3, 6, 10, 15 and 21.

Like in triangular numbers we can come up with the formula that we can use to find the value of a certain number in the sequence. Now let us write the formula of the sequence.

Writing the Formula of the n th Term in the Sequence

As you learned in Basic Processes of Algebra in your Junior Certificate Programme, we can also use the letters to represent the terms in the sequence. The position of the term is shown by writing the subscript. The sequence shown below shows the letters representing the terms and their respective positions in the sequence

This subscript represents the position of the term in the sequence

$a_1, a_2, a_3, \dots, a_n$ where, a s are terms in the sequence. Whereas,

- a_1 is the first term,
- a_2 is the second term,
- a_3 is the third term and so on,
- a_n is the nth term.

The nth term of a sequence is a term at a given position in a sequence. It is usually expressed as a_n where “a” is the term and “n” is the position of the term.

Let us look at the following example together. This example will help you to understand how to write a formula of the sequence into simple algebraic statements. Algebraic statements are mathematical sentences in which letters of alphabet are used to represent the unknown numbers.

Example 1: Find the formula for the nth term of the sequence

4, 9, 16, 25...

Solution:

Inspecting the numbers, we find that the numbers are square numbers. Therefore, we can express these terms as follows:

First Term in the Sequence

The term 4 is in the first position and can be expressed as 2×2 or in short as 2^2 by using 1 which represents the first position of the term 4, and making it to be part of 2. You should remember that 2 is the square root of 4. We can therefore express 2 as $(1 + 1)$.

Thus $4 = 2^2 = (1 + 1)^2$

Let us look at the second term.

Second Term in the Sequence

The second term in the sequence is 9. Since 9 is a square number its square root is 3. Therefore, 9 can be expressed as 3×3 or in short as 3^2 .

Since the term 9 occupies the second position in the sequence, then we will make 2 to be part of 3 which can be expressed as $(2 + 1)$.

We can therefore express 9 as:

$$9 = 3^2 = (2 + 1)^2$$

Let us move to the third term in the sequence.

Third Term in the Sequence

The third term in the sequence is 16. We can express 16 as 4×4 or in short as 4^2 . Considering the position of 16 in the sequence, which is 3, we will then express 4 as $(3 + 1)$.

$$\text{Therefore, } 16 = 4^2 = (3 + 1)^2.$$

Let us move to the fourth term in the sequence.

Fourth Term in the Sequence

As we can see from the sequence 4, 9, 16, 25, the fourth term is 25. The term 25 can be expressed as 5×5 or in short as 5^2 . The number 5 which is the square root of 25 can be written as $(4 + 1)$. Therefore, $25 = 5^2 = (4 + 1)^2$.

We can summarise the terms of the sequence 4, 9, 16, 25 in the table below.

1 st term	2 nd term	3 rd term	4 th term	n th term
4	9	16	25	a_n
2^2	3^2	4^2	5^2	
$(1 + 1)^2 = 4$	$(2 + 1)^2 = 9$	$(3 + 1)^2 = 16$	$(4 + 1)^2 = 25$	$(n + 1)^2$

Therefore, the formula for the nth term in the sequence 4, 9, 16, 25 is

$$a_n = (n + 1)^2.$$

You should do the following activity. The activity will help you understand how to write the formula for the sequence. The activity is similar to the example we have just done.



Activity 8

The first four terms of a sequence are

0, 3, 8, 15...

Write down an expression in terms of n for n th term of this sequence.

Write down your work and answer in the space provided below.



Feedback

Feedback

We will follow the following steps to write down an expression in terms of “ n ” for n th term of this sequence

Step 1: We should identify the rule that forms this sequence. By inspection we can see that all the terms are 1 less than square numbers.

Step 2: We should write an expression of the numbers with respect to their position in the sequence.

The terms we have in the sequence are

The first term = 0;

The second term = 3; and

The third term = 15.

We should come up with the formula of finding the values of these terms.

First term is 0

To find the expression for the first term “0”, we have to study first how the other terms in the sequence are found.

Let us begin with the second term.

Second term is 3

The term 3 can be expressed as

$$3 = 4 - 1;$$

Using 2 to represent second position of the term 3, we can make 2 as part of 3 as follows:

$$3 = 2^2 - 1$$

Since 4 can be expressed as 2^2 .

Let us move to the third term.

Third term is 8

The term 8 can be expressed as

$$8 = 9 - 1$$

We will use the number 3 which corresponds to the position of 8 and express the term 8 in expanded form as follows:

$$8 = 3^2 - 1$$

Remember 9 can be written as 3^2

Let us move to the fourth term.

Fourth term is 15

Using the pattern we have developed, we can express the term 15

$$15 = 16 - 1$$

Since the term 15 occupies the fourth position in the sequence, then we will square 4 and then subtract 1 from the answer as shown below.

$$15 = 4^2 - 1$$

Let us now consider the first term.

First term is 0

Now let us write expression of the first term. Since we are squaring the number of the position of the term and the subtract 1, the zero can be written as follows:

$$0 = 1^2 - 1$$

Now that we have identified the pattern that forms the sequence, we can write down an expression in terms of n for n th term of this sequence.

The n th term

Like in other terms in the sequence 0, 3, 8, 15... we will follow the same procedures to express the term occupying the position represented by the letter " n ".

Therefore, the n th term is $n^2 - 1$. We will finally write our answer as:

$$a_n = n^2 - 1$$

We can summarise our work in the table below.

1 st term	2 nd term	3 rd term	4 th term	n th term
0	3	8	15	a_n
$1^2 - 1$	$2^2 - 1$	$3^2 - 1$	$4^2 - 1$	$n^2 - 1$
$1 - 1 = 0$	$4 - 1 = 3$	$9 - 1 = 8$	$16 - 1 = 15$	$n^2 - 1$

We have squared the position number of the term and subtracted by 1

So far we have learned how to write the formula for the n th term of the given sequence. Now we will use the formula to find the value of the given n th term.

Finding the Value of the n th Term

When the position of the term and the formula are given then we can find the value of the term by substituting the letter in the formula by the position number. Substitution of numbers for letters is not new to you in the sense you learned it in your Junior Certificate Programme. Let us do the following example together to revise what we know about substitution of numbers for letters.

Example 2: Let us find the 7th term in the sequence whose n th term is $3n - 1$.

Solution

In this example, we are required to find the value of the term that occupies the seventh position in the sequence. The formula we will use to find the 7th term in the sequence is:

$$a_n = 3n - 1, \text{ when } n = 7, \text{ then}$$

$$a_7 = 3 \times 7 - 1$$

$$= 21 - 1$$

$$= 20$$

We have substituted n for 7

Therefore, the 7th term is **20**.

So far we have learned how to write the formula for finding the value of n th term using the position the term occupies in the sequence. Now we will write the general formula for the n th term of the arithmetic sequence by using the first term and common difference.

Writing the General Formula of the n th Term of the Arithmetic Sequence

What if you wanted to find the fiftieth term in a sequence? It would take a long time to find the fiftieth term by repeatedly adding the common difference. However, you can find any term by writing an expression that describes the sequence.

Let us consider the sequence 1, 4, 7, 10, 13...

We would go through the following process to arrive at an expression that describes the sequence.

Finding the common difference

The common difference of this sequence is

$$4 - 1 = 3$$

We then find the difference of the next two consecutive terms of the sequence to see if we will get the same answer.

$$7 - 4 = 3$$

Therefore, the common difference is 3

Finding the Second Term

In order to find the second term in the sequence we have to add the first term to the common difference of the sequence.

Second term = first term + common difference

$$\begin{aligned} a_2 &= 1 + 3 \\ &= 4 \end{aligned}$$

Finding the Third Term

In order to find the third term we will have to multiply the common difference by two and then add the first term as shown below.

Third term = first term + (common difference \times 2)

$$\begin{aligned} a_3 &= 1 + (3 \times 2) \\ &= 1 + 6 \\ &= 7 \end{aligned}$$

Take note of the way the second part of the expression is written.

Finding the Fourth Term

In order to find the fourth term we will have to multiply the common difference by three and then add the first term.

Fourth term = first term + (common difference \times 3)

$$\begin{aligned} a_4 &= 1 + (3 \times 3) \\ &= 1 + 9 \\ &= 10 \end{aligned}$$

Only one number inside the bracket is keep on changing as we move to another term.

Finding the Fifth Term

As we have done with the other terms we can express the fifth term as follows:

Fifth term = first term + (common difference \times 4)

$$\begin{aligned}
 a_5 &= 1 + (3 \times 4) \\
 &= 1 + 12 \\
 &= 13
 \end{aligned}$$

From our work above, we have expressed the terms of the sequence 1, 4, 7, 10, 13... as

The first term: $a_1 = 1$

The second term: $a_2 = 1 + (3 \times 1) = 4$

The third term: $a_3 = 1 + (3 \times 2) = 7$

The fourth term: $a_4 = 1 + (3 \times 3) = 10$

The fifth term: $a_5 = 1 + (3 \times 4) = 13$

From these we should identify the rule or pattern that is emerging from these expressions. The above work has been summarised in the table that forms part of the activity below.



Activity 9

We have added two columns to the table for you to complete. Writing down the 6th term and 7th term in the sequence, and their expressions using the knowledge you have gained so far.

1 st term	2 nd term	3 rd term	4 th term	5 th term	6 th term	7 th term
1	4	7	10	13		
	$1 + 3(1)$	$1 + 3(2)$	$1 + 3(3)$	$1 + 3(4)$		

Write your work and answer in the space provided.



Feedback

Feedback

When you go through the table you should be aware that each term has the first term, 1, and the common difference, 3. You should also be aware that the common difference is multiplied by the position number of the term less 1.

1 st term	2 nd term	3 rd term	4 th term	5 th term	6 th term	7 th term
1	4	7	10	13	16	19
	$1 + 3(1)$	$1 + 3(2)$	$1 + 3(3)$	$1 + 3(4)$	$1 + 3(5)$	$1 + 3(6)$

First term used in the formula

Common difference used in the formula

Position number (n) subtract 1

From what we have discussed above we can come up with the expression that we can use to find the value of the nth term.

First term + common difference \times position number less one

$$\text{First term} + \text{common difference} \times (n - 1)$$

Using this expression, we can write the general arithmetic progression as:

1 st term	2 nd term	3 rd term	4 th term	nth term
A	$a + d$	$a + 2d$	$a + 3d$	$a + (n - 1)d$

The sequence can as well be written as:

$$a, (a + d), (a + 2d), (a + 3d), \dots$$

Where

a = the first term,

d = the common difference,

You should also be aware that in the second term

The “d” in the 2nd term = $1 \times d$ or $(2 - 1) \times d$

The number 2 represents the second position of the term in the position.

The “d” in the 3rd term = $2 \times d$ or $(3 - 1) \times d$

The number 3 represents the third position of the term in the sequence.

The “d” in the 4th term = $3 \times d$ or $(4 - 1) \times d$

The number 4 represents the fourth position of the term in the sequence.

The pattern that is merging in finding the terms in the sequence is:

The value of 1st term = $a + (1 - 1) \times d = a + 0 \times d = a + 0 = a$

The value of 2nd term = $a + (2 - 1) \times d = a + 1 \times d = a + d$

The value of 3rd term = $a + (3 - 1) \times d = a + 2 \times d = a + 2d$

The value of 4th term = $a + (4 - 1) \times d = a + 3 \times d = a + 3d$

The value of nth term = $a + (n - 1) \times d$

From what we have just discussed, we can deduce that the nth term of this progression is given by the formula.

$$a_n = a + (n - 1) d,$$

Where: **a** = the first term,

d = the common difference,

n = the number that represents the position of the term

Now you should practice to use the formula we have just formulated by doing the following activity.



Activity 10

Find the tenth term of this sequence:

26, 38, 50, 6...

Write down your work and answer in the space below.

Feedback



Feedback

We can follow the following steps to find the 10th term of the sequence: 26, 38, 50, 6...

Step 1: We have to find the common difference by subtracting the second term from the first term.

$$38 - 26 = 12$$

We should also subtract the third term from the second term to see if we will still get the same answer.

$$50 - 38 = 12$$

Therefore, we can now write an expression. The first term is 26, the common difference is 12 and the n th term is 10.

$$a_n = a + (n - 1) d$$

$$a_{10} = 26 + 12 \times (10 - 1)$$

$$= 26 + 12 \times 9$$

$$= 26 + 108$$

$$= 134$$

We substitute the values of the first term, common difference and the n th term in the formula

We first find the product before finding the sum. Remember to use BODMAS you learned in Grade 10

Therefore, the tenth term = **134**.

In grade 10 you learned that BODMAS is an acronym which means the word that it formed from the initial letters of Brackets Of Division, Multiplication, Addition and Subtraction.

We have come to the end of topic 2. You should read the topic summary that follows.

Topic 2 Summary



Summary

In this topic, **Topic 2: The nth Term of a Sequence**, you learned how to write a formula that we could use to find the nth term of the sequence. We discussed two ways of writing the formula. The first one involves the use of the position of the term in the sequence as shown below: $a_1, a_2, a_3, \dots a_n$; where 1, 2, 3, and n are numbers representing the position of the terms in the sequence.

You also learned how to find the value of any nth term of the arithmetic progression. The formula to find the value of the nth term of the arithmetic progression was given as:

$$a_n = a + (n - 1) d,$$

where a = the first term

d = the common difference and

n = the position of the term.

You were also encouraged to answer all the questions in the activities provided in this topic. The activities were intended to help you assess how well you understood the content of the topic. You should now be well prepared to do the topic exercise which you will find at the end of the unit. If you do not do very well in the topic exercise, this means that you need to go over the material again.

Since this is the last topic, you should now read the unit summary that follows.

Unit Summary



Summary

Congratulation for having completed this last unit in your mathematics 11 course. Before you are given a chance to assess yourself on what you have learned in this unit by writing the tutor marked assignments you need to review briefly the entire unit.

In topic 1 you learned about sequences. You learned how to find a rule for the sequence and used it to continue the sequence. In addition you learned how to find the missing terms in the sequence. You also learned how to identify the patterns within and across different sequences and also how to find common differences of the arithmetic sequence and common ratio of geometric sequence.

In topic 2 you learned how to generalise sequences into simple algebraic statements. You learned how to use the general formula to find the n th term of arithmetic progression as well.

You were encouraged to assess your own progress by doing several activities and working out topic exercises after completing each topic. By comparing your work and answers to the model answers provided immediately after each activity you would have assessed your progress as you studied the different parts of the topics.

Now you should work out topic exercise 2 that is given after topic exercise 1 that comes soon after this topic. Follow the same procedure that was recommended for checking your answers for topic exercise 1.

This unit has a tutor-marked assignment. In the tutor marked assignment 5, you will be assessed on what you learned in units 13, 14 and 15. After you have completed working on the tutor marked assignment, you should send it to the college for marking.

There is also sixth assignment for you to do. This assignment is **an examination. It summarises your entire mathematics 11 course. You need to write it at the study centre. Therefore, you need to visit your study centre and arrange for your examination (sixth assignment).** Prepare yourself adequately for this examination. You are expected to write two papers.

Once more congratulations on reaching this important milestone of completing your study of the fifteen units of the grade 11 mathematics course. We hope that you enjoyed studying this course and that you will make good use of the skills and knowledge acquired. Lastly, we wish you all the best for your examinations (sixth assignment) and further studies.

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Topic Exercise 1



Assignment

Answer the following questions on a separate answer sheet.

- Find the next **three** terms of each sequence.
 - 1, 2, 4, 8, ____, ____, ____
 - $\frac{1}{3}, \frac{1}{9}, \frac{1}{27},$ ____, ____, ____
- Find the common difference of the following arithmetic sequences. Then find the next **two** terms in each sequence.
 - 2, 4, 6, 8, 10, ____, ____
 - 9, 3, -3, -9, ____, ____
 - 2, 2.5, 3, 3.5, ____, ____
- What are the first four terms in an arithmetic sequence with a common difference of $3\frac{1}{3}$ if the first term is 4?
- Find the common ratio of the following geometric sequences. Then write the next term.
 - 4, -16, -64, -256, ____
 - 243, 81, 27, ____
 - $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81},$ ____
- The sixth term of a geometric sequence is 81. The common ratio is $-\frac{1}{3}$. Find the first five terms.

Now compare your answers with those provided at the end of the unit.

Topic Exercise 2



Assignment

Work out the following problems on a separate answer paper.

1. Find a formula for the n th term of the following sequences:
 - (b) 1, 2, 3, 4, ...
 - (c) 2, 4, 6, 8, ...
 - (d) 4, 8, 12, 16, ...
2. (a) Find the first three terms of the sequence whose n th term is given by the formula $a_n = 3n - 5$.
(b) Find the 160th term of the sequence in (a).
1. Find the twentieth term in the sequence 18, 14, 10, 6, ...
2. Find the 102th term of the sequence 5, 13, 21, 29, 37, 45, ...
3. The fifth term of an arithmetic progression is 20 and common difference is -2. Find the first four terms.
4. The second term of an arithmetic progression is -4 and the 7th term is 11. Find the first term and the common difference.
5. Find the 10th terms of the formula
 - a) $a_n = n(n - 1)$
 - b) $a_n = \frac{n}{n + 1}$

Answers for Unit 15 Topic Exercises

Answers to Topic Exercise 1



Answers to Assignment

1. (a) A rule for sequence: Multiply the preceding term by 2
The next three terms of the sequence are **16, 32** and **64**.

- (b) A rule is multiply the preceding term by $\frac{1}{3}$.

The next three terms are: $\left(\frac{1}{27} \times \frac{1}{3}\right) = \frac{1}{81}$

$$\left(\frac{1}{81} \times \frac{1}{3}\right) = \frac{1}{243}$$

$$\left(\frac{1}{243} \times \frac{1}{3}\right) = \frac{1}{729}$$

2. (a) Common difference = **2**;

The next two terms are **12** ($= 10 + 2$) and **14** ($= 12 + 2$)

- (b) Common difference = **-6**;

The next two terms are **-15** ($= -9 - 6$) and **-21** ($= -15 - 6$)

- (c) Common difference = **0.5**;

The next two terms are **4** ($= 3.5 + 0.5$) and **4.5** ($= 4.5 + 0.5$)

3. The first four terms are 4, $7\frac{1}{3}$, $10\frac{2}{3}$ and 14.

4. (a) Common ratio = $\frac{\text{second term}}{\text{first term}} = \frac{-16}{-4} = 4$

The next term is **-1024** ($= -256 \times 4 = -1024$)

- (b) Common ratio = $\frac{\text{second term}}{\text{first term}} = \frac{81}{243} = \frac{1}{3}$

The next term is **9** ($= 27 \times \frac{1}{3} = 9$)

- (c) Common ratio = $\frac{\text{second term}}{\text{first term}} = \frac{1}{9} \div \frac{1}{3} = \frac{1}{9} \times \frac{3}{1} = \frac{1}{3}$

The next term is $\frac{1}{243}$ ($= \frac{1}{81} \times \frac{1}{3} = \frac{1}{243}$)

$$5. \frac{\text{6th term}}{\text{5th term}} = -\frac{1}{3}$$

$$\begin{aligned} 5^{\text{th}} \text{ term} &= 81 \div -\frac{1}{3} \\ &= 81 \times -3 \\ &= \mathbf{-243} \end{aligned}$$

$$\begin{aligned} 4^{\text{th}} \text{ term} &= -243 \times -3 \\ &= \mathbf{729} \end{aligned}$$

$$\begin{aligned} 3^{\text{rd}} \text{ term} &= 729 \times -3 \\ &= \mathbf{-2187} \end{aligned}$$

$$\begin{aligned} 2^{\text{nd}} \text{ term} &= -2187 \times -3 \\ &= \mathbf{6561} \end{aligned}$$

$$\begin{aligned} 1^{\text{st}} \text{ term} &= 6561 \times -3 \\ &= \mathbf{-19683} \end{aligned}$$

We hope that after comparing your answers with the model answers provided above, you might have got all the answers correct. Congratulations! If not then I suggest you try the questions you got wrong again. You may be required to read through the section again in order to gain more understanding.

Answers to Topic Exercise 2



Answers to Assignment

1. (a) Each term of the sequence is obtained by multiply its position by 1.

$$\text{First term} = 1 \times 1 = 1$$

$$\text{Second term} = 2 \times 1 = 2$$

$$\text{Third term} = 3 \times 1 = 3$$

$$\text{Fourth term} = 4 \times 1 = 4$$

$$\text{nth term} = n \times 1 = n$$

Therefore, the formula for nth term is $a_n = n$

- (b) Each term of the sequence is obtained by multiply its position by 2.

$$\text{First term} = 1 \times 2 = 2$$

$$\text{Second term} = 2 \times 2 = 4$$

$$\text{Third term} = 3 \times 2 = 6$$

$$\text{nth term} = n \times 2 = 2n$$

Therefore, the formula for nth term is $a_n = 2n$.

- (c) Each term of the sequence is obtained by multiply its position by 4.

$$\text{First term} = 1 \times 4 = 4$$

$$\text{Second term} = 2 \times 4 = 8$$

$$\text{Third term} = 3 \times 4 = 12$$

$$\text{nth term} = n \times 4 = 4n$$

Therefore, the formula for nth term is $a_n = 4n$.

2. (a) $a_n = 3n - 5$

$$\text{When } n = 1, \text{ then } a_1 = 3 \times 1 - 5 = 3 - 5 = -2.$$

$$\text{When } n = 2, \text{ then } a_2 = 3 \times 2 - 5 = 6 - 5 = 1$$

$$\text{When } n = 3, \text{ then } a_3 = 3 \times 3 - 5 = 9 - 5 = 4$$

Therefore, the three terms are -2, 1 and 4.

$$\begin{aligned} \text{(a) } a_{160} &= 3 \times 160 - 5 \\ &= 480 - 5 \end{aligned}$$

$$= 475$$

3. $a_n = a_1 + (n - 1)d$, when $n = 20$, $d = -4$, $a_1 = 18$

$$a_{20} = 18 + (20 - 1) \times -4$$

$$= 18 + 19 \times -4$$

$$= 18 - 76$$

$$= -58$$

Therefore, the twentieth term is - 58.

4. $a_n = a_1 + (n - 1)d$, when $n = 102$, $d = 8$, $a_1 = 5$

$$a_{102} = 5 + (102 - 1) \times 8$$

$$= 5 + 101 \times 8$$

$$= 5 + 808$$

$$= 813$$

Therefore, the 102th term is 813.

5. $a_n = a_1 + (n - 1)d$, when $n = 5$, $d = -2$, $a_5 = 20$

$$20 = a_1 + (5 - 1) \times -2$$

$$20 = a_1 + 4 \times -2$$

$$20 = a_1 - 8$$

$$a_1 = 20 + 8$$

$$= 28$$

Therefore, the first four terms are 28, 26, 24 and 22.

6. $a_n = a_1 + (n - 1)d$, when $n = 2$, $d = ?$, $a_2 = -4$

$$-4 = a_1 + (2 - 1) \times d$$

$$-4 = a_1 + 1 \times d$$

$$-4 = a_1 + d$$

$$a_1 = -4 - d$$

When $n = 7$, $d = ?$ $a_7 = 11$

$$11 = a_1 + (7 - 1) \times d$$

$$11 = a_1 + 6 \times d$$

$$11 = a_1 + 6d$$

$$a_1 = 11 - 6d$$

Therefore,

$$-4 - d = 11 - 6d$$

$$6d - d = 11 + 4$$

$$5d = 15$$

$$d = 15 \div 5$$

$$= 3$$

Substitute 3 for d in $a_1 = 11 - 6d$

$$a_1 = 11 - 6(3)$$

$$a_1 = 11 - 18$$

$$a_1 = -7$$

Therefore, the first term is -7 and the common difference is 3.

7. (a) $a_n = n(n - 1)$

$$a_{10} = 10(10 - 1)$$

$$= 10 \times 9$$

$$= 90$$

(b) $a_n = \frac{n}{n + 1}$

$$a_{10} = \frac{10}{10 + 1}$$

$$= \frac{10}{11}$$

We hope that after checking your answers against the model answers provided in the feedback above you might have got all the answers correct. This shows that you worked hard and understood the topic very well. Congratulations! If you got some of the answers wrong then I suggest you go through the calculations leading to the answers in the feedback and note down where you made mistakes in your work. You should as well practice to work out questions that you performed poorly.

Now you should do tutor marked assignment 5 for units 13, 14 and 15. After you have completed answering the assignment, you then send the answered worksheets to the college for marking.

After you have completed the tutor marked assignment, you should then be ready to do the final test based on the entire mathematics 11. You were informed about this in the unit summary above. Remember that you need to visit your study

centre and arrange for your examination (sixth assignment). Prepare yourself adequately for this examination. You are expected to write two papers. Good luck!

Tutor-marked Assignment 5 (Units 13, 14 & 15)



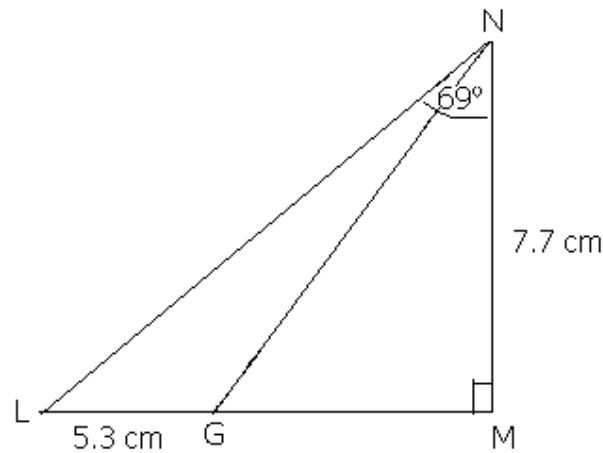
Assessment

Answer the following questions on a separate answer sheet.

1. A wall is “h” metres high. A ladder, leaning from the top of the wall to the ground is 15 metres long. The ladder makes an angle of elevation of 54° with the horizontal ground. Find:

- (a) The distance of the foot of ladder from the bottom of the wall. [2]
 (b) The value of “h”. [2]

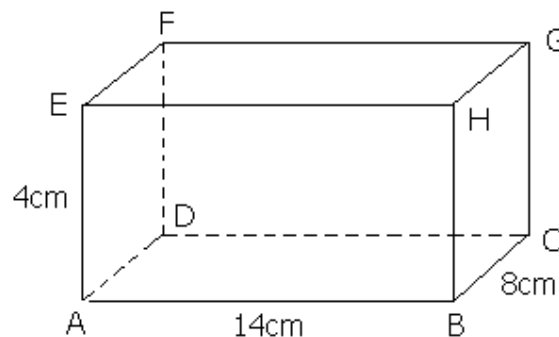
2.



In triangle LMN above, $MN = 7.7$ cm, $LG = 5.3$ cm and $\angle LNM = 69^\circ$. Calculate:

- (a) LM [2]
 (b) LN [2]
 (c) $\angle GNM$ [3]

3.



A cuboid ABCDEFGH has length $AB = 14$ cm, breadth $BC = 8$ cm and height $AE = 4$ cm, calculate

- (a) DE [2]
 (b) $\angle AED$ [2]
 (c) AG [3]
 (d) $\angle AGC$ [2]

4. Find the mode, mean, and median of the following numbers: 9, 11, 7, 11, 7, 8, 12, 7, 9. [4]

5. Find x , if the mean of the numbers 4, 5, x , 8, 9 and 3 is 8. [2]

6. The table below shows the mark obtained in an English examination:

Mark	Frequency
30-39	3
40-49	4
50-59	10
60-69	5
70-79	7
80-89	4
90-99	2

- (a) If the pass mark was 50%, how many learners would pass? [1]
 (b) If $51\frac{3}{7}\%$ of the learners passed, what was the pass mark? [2]
 (c) From the table calculate the mean mark, correct to 1 decimal place [3]

7. Find the common difference of the following arithmetic sequences. Then find the next two terms in each sequence.

(d) 0, 5, 10, 15, ... [2]

(e) 30, 20, 10, 0, ... [2]

8. Find the common ratio of the following geometric sequences. Then write the next term.

(d) 8, 12, 18, 27, ... [2]

(e) 8, 4, 2, 1, ... [2]

9. What is the thirtieth term in the sequence 1.5, 4, 6.5, 9, .. [3]

10. The second term of an arithmetic progression is - 4 and the 7th term is 11. Find the first term and the common difference [6]

Total marks: 50

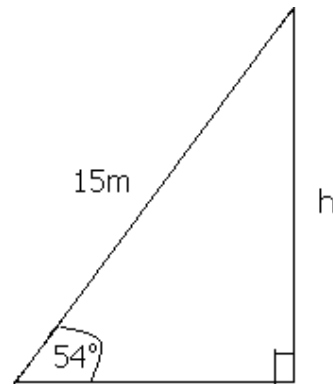
Model Answers (At college)



Answers to Assessment

Answer the following questions on a separate answer sheet.

1.



$$\begin{aligned}
 \text{(a)} \quad \cos 54^\circ &= \frac{x}{15} \\
 h &= 15 \times \cos 54^\circ \\
 &= 15 \times 0.588 \\
 &= 8.817 \\
 \text{Distance} &= 8.817\text{m} \qquad [2]
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \sin 54^\circ &= \frac{h}{15} \\
 h &= 15 \times \sin 54^\circ \\
 &= 15 \times 0.809 \\
 &= 12.13\text{m} \\
 \text{Height} &= 12.13\text{m} \qquad [2]
 \end{aligned}$$

$$\begin{aligned}
 \text{2. (a)} \quad \tan 69^\circ &= \frac{LM}{7.7} \\
 LM &= 7.7 \times \tan 69^\circ \\
 &= 7.7 \times 2.9 \\
 &= \mathbf{20.06} \qquad [2]
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \cos 69^\circ &= \frac{7.7}{LN} \\
 LN &= \frac{7.7}{\cos 69^\circ} \\
 &= \frac{7.7}{0.358} \\
 &= 21.5 \text{ cm} \qquad [2]
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad GM &= LM - LG \\
 &= 20.06 - 5.3
 \end{aligned}$$

$$\begin{aligned}
 &= 14.76 \\
 \text{Tan } \angle \text{GNM} &= \frac{14.76}{7.7} \\
 &= 1.916 \\
 &= \mathbf{62.4^\circ}
 \end{aligned}
 \tag{3}$$

3

$$\begin{aligned}
 \text{(a) } DE &= \sqrt{AD^2 + AE^2} \\
 &= \sqrt{8^2 + 4^2} \\
 &= \sqrt{64 + 16} \\
 &= \sqrt{80} \\
 &= \mathbf{8.94 \text{ cm}}
 \end{aligned}
 \tag{2}$$

$$\begin{aligned}
 \text{(b) } \text{Tan } \angle \text{AED} &= \frac{8 \text{ cm}}{4 \text{ cm}} \\
 &= 2 \\
 &= \mathbf{63.4^\circ}
 \end{aligned}
 \tag{2}$$

$$\begin{aligned}
 \text{(c) } AC &= \sqrt{BC^2 + AB^2} \\
 &= \sqrt{8^2 + 14^2} \\
 &= \sqrt{64 + 196} \\
 &= \sqrt{260} \\
 &= 16.12 \text{ CM}
 \end{aligned}$$

$$\begin{aligned}
 AD &= \sqrt{AC^2 + CG^2} \\
 &= \sqrt{16.1^2 + 4^2} \\
 &= \sqrt{260 + 16} \\
 &= \sqrt{276} \\
 &= \mathbf{16.61 \text{ cm}}
 \end{aligned}
 \tag{3}$$

$$\begin{aligned}
 \text{(d) } \text{Tan } \angle \text{AGC} &= \frac{16.12 \text{ cm}}{4 \text{ cm}} \\
 &= 4.03 \\
 &= \mathbf{76.06^\circ}
 \end{aligned}
 \tag{2}$$

4. Arranging the data in ascending order = 7, 7, 7, 8, 9, 9, 11, 11, 12

$$\text{Mode} = 7 \tag{1}$$

$$\begin{aligned}
 \text{Mean} &= \frac{\sum x}{n} = \frac{9 + 11 + 7 + 11 + 8 + 12 + 7 + 9}{9} \\
 &= \frac{81}{9} \\
 &= 9
 \end{aligned}
 \tag{2}$$

$$\text{Median} = 9 \tag{1}$$

$$5. \frac{4 + 5 + x + 8 + 9 + 3}{6} = 6$$

$$\frac{29+x}{6} = 6$$

$$29 + x = 36$$

$$x = 36 - 29$$

$$x = 7 \quad [2]$$

6.

(a) Number of learners passed = $10 + 5 + 7 + 2$
= **28 learners** [1]

(b) Total number of learners = 35
Therefore, $51\frac{3}{7}\%$ of 35 = $\frac{155}{700} \times 35 = 18$
Therefore, pass mark = **60%** [2]

(c)

Mark	Frequency	Mid-mark (x)	F(x)
30-39	3	35	105
40-49	4	45	180
50-59	10	55	550
60-69	5	65	325
70-79	7	75	525
80-89	4	85	340
90-99	2	95	190
Total	35		2,215

$$\text{Mean mark} = \frac{\sum fx}{fx} = \frac{2215}{35} = 63.2\% \quad [3]$$

7. (a) 0, 5, 10, 15, ...

Common difference = **5**;Since $5 - 0 = 5$ or $10 - 5 = 5$.The next two terms are **20** and **25** [2]Since $20 = 15 + 5$ and $25 = 20 + 5$

(b) 30, 20, 10, 0, ...

Common difference = **-10**;

Since $20 - 30 = -10$ or $10 - 20 = -10$

The next two terms are **-10** and **-20** [2]

Since $-10 = 0 - 10$ and $-20 = -10 - 10$

8. (a) 8, 12, 18, 27, ...

Common ratio = $\frac{\text{second term}}{\text{first term}} = \frac{12}{8} = \frac{3}{2}$

The next term is $40\frac{1}{2}$ [2]

Since $27 \times \frac{3}{2} = \frac{27 \times 3}{2} = \frac{81}{2} = 40\frac{1}{2}$

(b) 8, 4, 2, 1, ...

Common ratio = $\frac{\text{second term}}{\text{first term}} = \frac{4}{8} = \frac{1}{2}$

The next term is $\frac{1}{2}$ [2]

Since $\frac{1}{2} = 1 \times \frac{1}{2}$

9. $a_n = a_1 + (n - 1)d$, when $n = 30$, $d = 2.5$, $a_1 = 1.5$

$$\begin{aligned} a_{30} &= 1.5 + (30 - 1) \times 2.5 \\ &= 1.5 + 29 \times 2.5 \\ &= 1.5 + 72.5 \\ &= 74 \end{aligned}$$

Therefore, the thirtieth term is **74**. [3]

10. $a_n = a_1 + (n - 1)d$, when $n = 2$, $d = ?$, $a_2 = -4$

$$-4 = a_1 + (2 - 1) \times d$$

$$-4 = a_1 + 1 \times d$$

$$-4 = a_1 + d$$

$$a_1 = -4 - d \quad [2]$$

When $n = 7$, $d = ?$ $a_7 = 11$

$$11 = a_1 + (7 - 1) \times d$$

$$11 = a_1 + 6 \times d$$

$$11 = a_1 + 6d$$

$$a_1 = 11 - 6d \quad [2]$$

Therefore,

$$-4 - d = 11 - 6d$$

$$6d - d = 11 + 4$$

$$5d = 15$$

$$d = 15 \div 5$$

$$= 3$$

[1]

Substitute 3 for d in $a_1 = 11 - 6d$

$$a_1 = 11 - 6(3)$$

$$a_1 = 11 - 18$$

$$a_1 = -7$$

[1]

Therefore, the first term is -7 and the common difference is 3.

Total marks: 50

Assignment – End of Grade 11

Mathematics Course Test: Paper 1



Assignment 6

This assignment should be written at a study centre under the supervision of a tutor

Time: 2 hours

Write your details below.

Name _____ Study Centre _____

Student Number: _____ Date: _____

Instructions to candidates

There are 15 questions in this paper. Answer all the questions by writing down your answers in the spaces provided on each question. You can use the spaces provided for your calculations.

-
1. The actual size of Zambia is square kilometers is 752 614.
Express this number
(a) to the nearest 1000 km²
(b) in standard form correct to 2 significant figures.

Answer: (a) [1]

(b) [2]

-
2. (a) If $f(x) = 3x - 2$, find the value of x given that $f(x) = -11$
(b) A relation is defined with range $\{4, 5, 6\}$. Find the domain if the relation is “plus 3 is”.

Answer: (a) [2]

(b)..... [1]

-
3. Solve the equation $1 - 5x - 2x^2 = 0$, giving your answer correct to 3 significant figures.

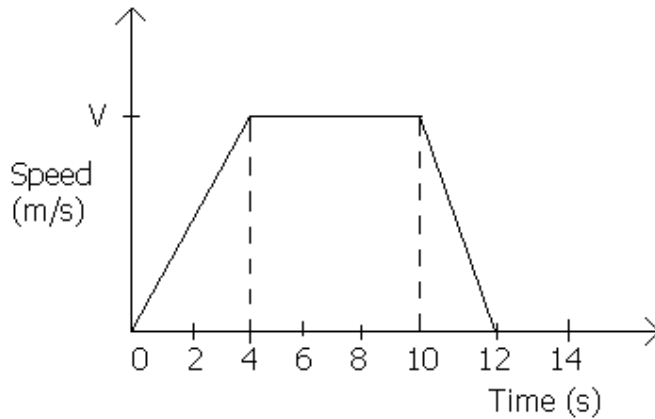
Answer: [5]

4. (a) A length of wire can be cut in 6 pieces each 30 cm long. How many pieces each 18 cm long, can be cut from this. [2]
 (b) If $y + 4$ varies directly as x and $y = 8$ when $x = 12$, find the value of y when $x = 7$.

Answer: (a) [2]

(b) [2]

5. The diagram below shows the speed-time graph of a car which starts from rest and increases its speed at a constant rate for 4 seconds and then travels at a constant speed of v m/s for a further 6 seconds before it comes to a halt in 2 seconds.



Given that the car travelled 80 meters in the first 4 seconds, calculate the:

- (a) Maximum speed, v , the car reached
 (b) Acceleration in the last 2 seconds,
 (c) Retardation in the last 2 seconds,
 (d) Total distance travelled in the period of 12 seconds.

Answer: (a) [2]

(b) [1]

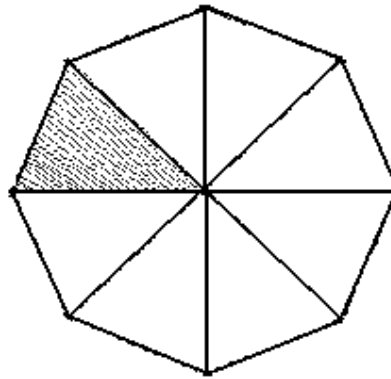
(c) [1]

(d) [2]

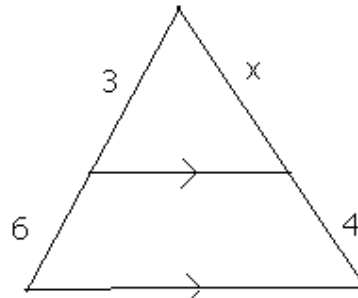
6. (a) How many lines of symmetry does a regular nonagon have?
 (b) The diagram below shows a regular octagon. Shade one more triangle of the octagon so that the diagram has rotational symmetry of order 2.

Answer: (a) [1]

(b) [1]

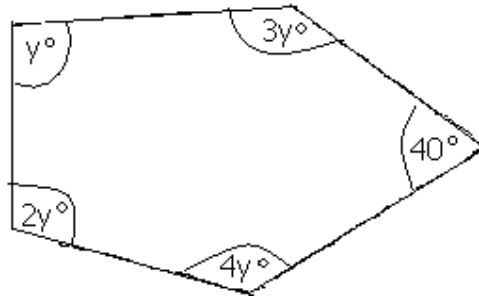


7. Calculate the value of x in the following triangles.



Answer $x =$ [2]

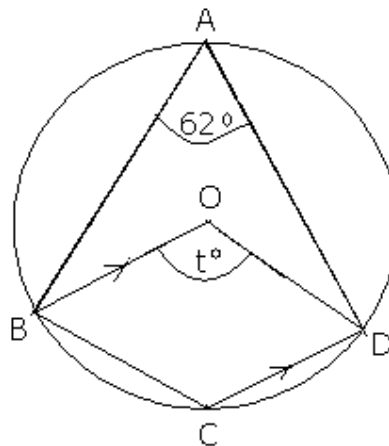
8. (a) Two similar triangles have corresponding sides of length 3 cm and 5 cm. Find the ratio of their areas.
 (b) The five angles of the pentagon below are y° , 40° , $2y^\circ$, $3y^\circ$ and $4y^\circ$. Calculate the value of y .



Answer (a)..... [1]

(b) [2]

9. In the diagram below, A, B, C, and D lie on the circumference of the circle, center O, BO is parallel to CD and $\angle BAD = 62^\circ$.



- Calculate
 (a) Angle t
 (b) $\angle BCD$
 (c) $\angle OBC$

Answer: (a) $t =$ [1]

(b) $\angle BCD$ [1]

(c) $\angle OBC$ [2]

10. If any interior angle of a regular polygon is x , the exterior angle is $\left(\frac{x-36^\circ}{3}\right)$ Find (i) the value of x (ii) the number of

sides in the polygon.

Answer: (i) [2]

(ii) [2]

11. The circumference of a circle is 56 cm. Find the radius of the circle to 2 decimal places. [take π to be 3.14]

Answer: [2]

12. C, D and E are points on level ground as shown in the diagram below.

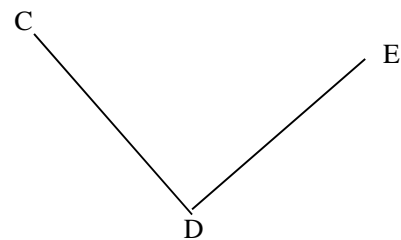
(a) Construct the locus of points which are:

(i) equidistant from CD and DE,

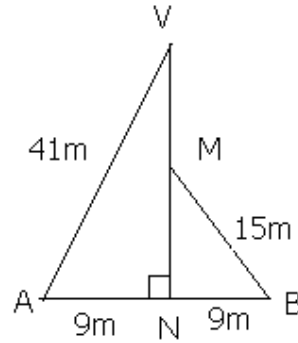
(ii) equidistant from C and E.

(b) Mark the point T which is equidistant from CD and DE and is also equidistant from C and E.

[5]



13. In the diagram below, ANB is a horizontal line and VN is a vertical mast supported by two stays AV and BM. Given that AV = 41 m, BM = 15 m and AN = NB = 9 m, calculate the length of MV.



Answer: [3]

14. Find the median if the mean of 10, 8, 7, 12, x, 6, 10 and 4 is 8.

Answer: [2]

15. (a) Write down the 7th term of the sequence 4, 7, 10, 13, ...
 (b) The fifth term of an arithmetic progression is 20 and the common difference is -2. Find the first four terms.

Answer: (a) [2]

(b) [2]

Total = **50 marks**

Assignment – End of Grade 11

Mathematics Course Test: Paper 2



Assignment 6

Time: 2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

Write your name, study centre and student number in the spaces provided on the separate answer paper/answer booklet.

Write your answers and work on the separate answer paper provided.

Show all your work on the same page that contains the rest of the answer.

Omission of essential working will result in loss of marks.

**SILENT NON PROGRAMMABLE CALCULATORS /
MATHEMATICAL TABLES MAY BE USED.**

INFORMATION FOR THE CANDIDATES

The number of marks is given in brackets [] at the end of each question or part question.

You are expected to use mathematical tables / or calculators where necessary to evaluate explicit numerical expressions.

If the degree of accuracy is not specified in the question, and if the answer is not exact, give the answer to three significant figures. Give answers in degrees to one decimal place.

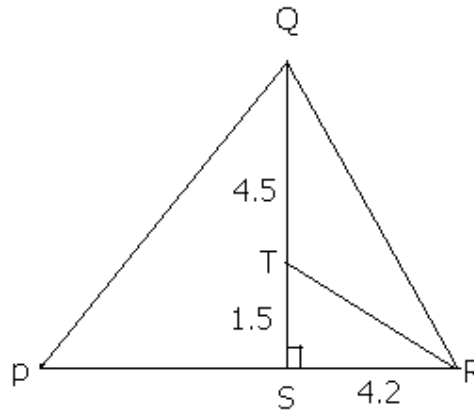
-
1. (a) A piece of wood is 20.8m long. If a 0.01355 m piece is sawn off, find the length correct to three decimal places, of the remaining piece of wood. [2]

(b) The speed of light is 300,000,000 m/s. How far would light travel in two hours? Give your answer in kilometers and express it in scientific notation. [2]

(c) Solve the equation $2x^2 + 5x - 1 = 0$, giving your answer correct to two decimal places. [5]

 2. (a) Draw the graph of the function $f: x \rightarrow 3x - 5$, for the domain $0 \leq x \leq 2, x \in \mathbb{R}$. [5]

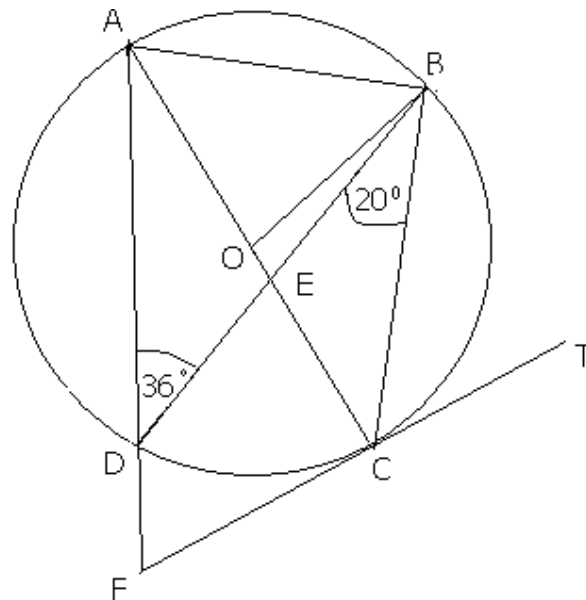
(b) The diagram below represents a roof structure at a construction site QTs is a vertical beam and PSR is a horizontal beam P, RT and RQ are supporting beams. $SR = 4.2\text{m}$, $ST = 1.5\text{m}$ $TQ = 4.5\text{m}$ and $\angle QPS = 62^\circ$.



Calculate

- (a) TR [2]
- (b) $\angle SQR$ [2]
- (c) PQ [2]

3. (a) Solve the equation $3x - 3x - 1 = 0$, giving your answers correct to 2 decimals. [5]
- (b)



In the diagram A, B, C and D are points on the circle, centre O, AC and BD intersect at E and the tangent to the circle at C meets AD produced at F. Given that $\angle ADB = 36^\circ$ and $\angle DBC = 20^\circ$.

Find

- (i) $\angle AOB$ [1]
- (ii) $\angle AED$ [1]

- (iii) $\angle AFC$ [1]
- (iv) $\angle OBD$ [1]
- (v) $\angle OAB$ [1]
- (vi) $\angle BCT$ [1]

Answer the whole of this question on a sheet of plain paper.

4. (a) Construct a triangle ABC in which $AB = 9\text{cm}$, $BC = 7\text{cm}$ and $\angle ABC = 40^\circ$. Measure and write down the length of AC. [2]
- (b) On the same diagram
- (i) Draw the locus of points which are 6cm from B [2]
 - (ii) Construct the locus of points equidistant from AC and AB [2]
- (c) P is a point inside triangle ABC such that it is 6cm from B, and equidistant from AB and AC. Label the point P. [1]
- (d) The point Q which lies inside the triangle ABC is such that its distance from B is less than 6cm and it is nearer to AC than to AB. Indicate clearly, by shading, the region in which Q must lie. [2]

Answer the Whole of this Question on a Sheet of Graph Paper.

5. The variables x and y are connected by the equation

$$y = \frac{x^2}{5} + \frac{15}{x} - 8$$

The table below shows some corresponding values of x and y.

The values of y are given to one decimal place where appropriate.

x	1	1.5	2	3	3.5	4	5	6	7
y	7.2	2.5	0.3	-1.2	-1.3	-1.1	0	1.7	p

- (a) Calculate the value of P correct to one decimal place. [1]
- (b) Using a scale of 2cm to 1 unit on each x-axis draw a horizontal axis for $0 \leq x \leq 8$ and a vertical axis of $-2 \leq y \leq 8$. On your axis plot the given points and join them with a smooth curve. [3]
- (c) By drawing a tangent, find the gradient of the curve at the point (3,-1.2) [2]
- (d) Use your graph to find the value of x for which $\frac{x^2}{5} + \frac{15}{x} = 8$ [2]
- (e) (i) On the same axis draw the graph of $y = 4 - x$ [2]

- (i) Use the graphs to solve the equation $\frac{x^2}{5} + \frac{15}{x} - 8 = 4 - x$ [2]

6. The floor of a sheep pen is in the shape of a quadrilateral ABCD. Each side of the quadrilateral is 3m long and angle ABD is 60° .
- (a) Using a scale of 2cm to represent 1 metre, draw the quadrilateral ABCD. [2]
- (b) Draw the axes of symmetry of quadrilateral ABCD. [1]
- (c) What is the special geometrical name of this quadrilateral? [1]
- (d) Measure the diagonal AC, giving your answer in meters, correct to one decimal place. [2]
- (e) Calculate the area of the floor in square metres giving your answer correct to 2 decimal places. [3]

7. (a) A survey was conducted in 2006 on the percentages of adults (aged 15 to 46) with HIV/AIDS in some towns in Zambia. The percentages for the highest six were as follows:

Lusaka	36 %
Ndola	23.5%
Chirundu	25%
Chipata	20%
Monze	20%
Livingstone	25%

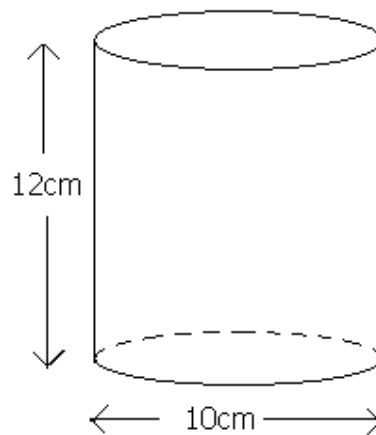
- (i) Draw a bar chart to illustrate the above information [3]
- (ii) Given that the number of those with HIV/AIDS in Chirundu is 8 000, calculate the total population in this town. [2]
- (iii) Given the population of Monze is 30 000 and that of Livingstone is 65 000, calculate the total number of those with HIV/AIDS in these two towns [3]
- (b) The table below shows the marks obtained by a grade 11 class for a mathematics test.

Mark	Frequency
45-49	2
50-54	3
55-59	5
60-64	3

65-69	0
70-74	10
75-79	1
80-84	4
85-89	2
90-94	4
95-99	1

- (a) How many learners sat for the test? [1]
 (b) Find the modal class interval. [1]
 (c) Find the mean mark. [2]

8. The diagram below represents a glass of water of uniform cross section and thickness.



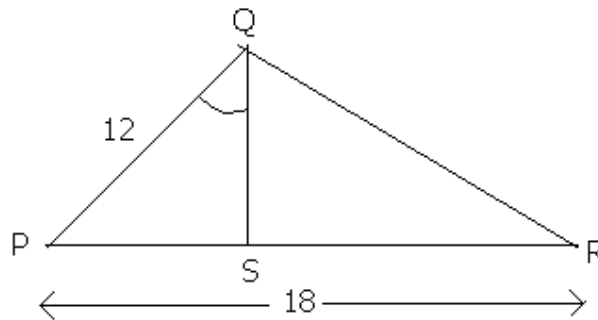
Given that its height is 12cm, diameter is 10cm and taking $\pi = 3.142$,

- (a) Calculate the outer surface area of the glass correct to 2 decimal places. [3]
 (b) Given that the thickness of the glass of water is 2mm, calculate the volume of water that the glass can hold when completely full, giving your answer correct to 2 decimal places. [4]

9. (a) Find in simplest form the ratio of 5 cm to 1.5 km. [2]
 (b) There is enough food in a camp to last 12 men for 10 days.
 How long would the food last, if there were 15 men? [2]
 (c) V varies directly as the square of x and inversely as y . Given
 that $v = 9$ when $x = 3$ and $y = 4$, find x when $v = 50$ and
 $y = 2$. [3]

10. (a) Two similar rectangles have corresponding sides in the ratio
 9:7. Find the ratio of their areas. [2]

(b)



In the triangle PQR, $PQ = 12$ cm, $PR = 18$ cm and S is a
 point on the side PR such that $\angle PQS = \angle PRQ$

- (i) Write down another pair of equal angles [1]
 (ii) Use similar triangles to calculate the length of PS [3]
 (iii) Given that the area of triangle PQS is 20 cm^2 ,
 Calculate the area of triangle PQR . [3]

11. (a) Find the value of 20th term of the sequence; 2, 5, 8, 11 ... [2]
 (b) Find a formula for the n th term of the following sequences [2]

Total = **100 marks**

Summative Assessment



Answers to Paper 1

These answers should be kept at study centres for tutors' use only

End of Grade 11 Mathematics Test Paper 1

1. (a) $752\ 614 = \underline{753\ 000}$ correct to the nearest 1 000 km² [1]

(b) $752\ 614 = 7.52\ 614 \times 10^5$
 $= \underline{7.2 \times 10^2}$ correct to 2 significant figures [2]

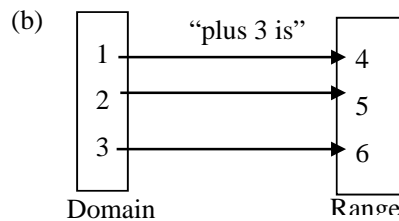
2. (a) $F(x) = 3x - 2$

$$-11 = 3x - 2$$

$$3x = 2 - 11$$

$$3x = -9$$

$$x = \underline{3}$$
 answer [2]



The domain is **{1, 2, 3}** Answer [1]

3. $X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$X = \frac{-(-5) \pm \sqrt{-5^2 - 4(-2)(1)}}{2(-2)}$$

$$X = \frac{5 \pm \sqrt{33}}{-4}$$

Either $x = \frac{5 + 5.745}{-4} = \underline{-2.69}$ correct to 3 significant figures or [1]

$x = \frac{5 - 5.745}{-4} = \underline{0.186}$ correct to 3 significant figures [1]

4. (a) Total length of wire = 30 cm × 6 pieces = 180 cm
 Number of pieces = 180cm ÷ 18 cm = **10 pieces**

(b) $y + 4 \propto x$

$$Y + 4 = k(x) \text{ when } y = 8 \text{ and } x = 12$$

$$8 + 4 = k \times 12$$

$$12 = 12k$$

$$k = 1$$

Thus; $y + 4 = x$, when $x = 7$ then

$$Y + 4 = 7$$

$$y = 7 - 4$$

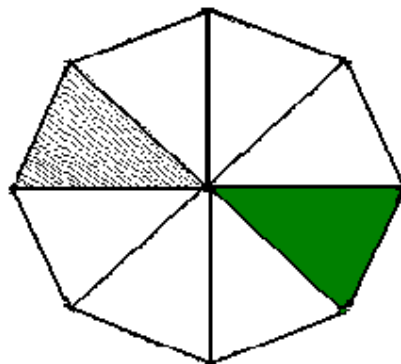
$$y = \underline{\mathbf{3 \text{ answer}}}$$
 [2]

5. (a) Speed = $\frac{\text{distance}}{\text{time}}$
 $= \frac{80 \text{ m}}{4 \text{ seconds}}$
 $= \underline{\mathbf{20 \text{ m/s}}}$ [1]

(b) Acceleration = $\frac{\text{speed}}{\text{time}}$
 $= \frac{\text{final speed} - \text{initial speed}}{\text{time taken}}$
 $= \frac{20 \text{ m/s} - 0 \text{ m/s}}{4 \text{ seconds}}$
 $= \frac{20 \text{ m/s}}{4 \text{ seconds}}$
 $= \underline{\mathbf{5 \text{ m/s}^2}}$ [1]

(c) Retardation = $\frac{\text{speed}}{\text{time}}$
 $= \frac{\text{final speed} - \text{initial speed}}{\text{time taken}}$
 $= \frac{0 \text{ m/s} - 20 \text{ m/s}}{2 \text{ seconds}}$
 $= \frac{-20 \text{ m/s}}{2 \text{ seconds}}$
 $= \underline{\mathbf{-10 \text{ m/s}^2}}$ [2]

6. (a) 9 lines of symmetry [1]
 (b) [1]



7. $3:9 = x:(x + 4)$
 $\frac{3}{9} = \frac{x}{x+4}$
 $3(x - 4) = 9x$
 $3x + 12 = 9x$

$$9x - 3x = 12$$

$$6x = 12$$

$$X = \underline{2 \text{ Answer}}$$

[2]

8. (a) $3:5 = \underline{9:25 \text{ answer}}$

[1]

$$(b) y^\circ + 40^\circ + 2y^\circ + 3y^\circ + 4y^\circ = 360^\circ$$

$$40^\circ + 10y^\circ = 360^\circ$$

$$10y^\circ = 360^\circ - 40^\circ$$

$$y^\circ = \frac{320^\circ}{10}$$

$$y^\circ = \underline{32^\circ \text{ Answer}}$$

[2]

9. (a) $t = 124^\circ$

[1]

(b) $\angle BCD = 118^\circ$

[1]

(C) $\angle OBC = 62^\circ$

[2]

10. (i) $x^\circ + \left(\frac{x-36}{3}\right)^\circ = 180^\circ$

$$3x^\circ + x - 36^\circ = 180^\circ \times 3$$

$$4x^\circ = 540^\circ$$

$$x = \underline{144^\circ \text{ answer}}$$

[2]

(ii) Number of sides = $\frac{360^\circ}{36} = \underline{10 \text{ sides Answer}}$ [2]

11. $C = 2\pi r$

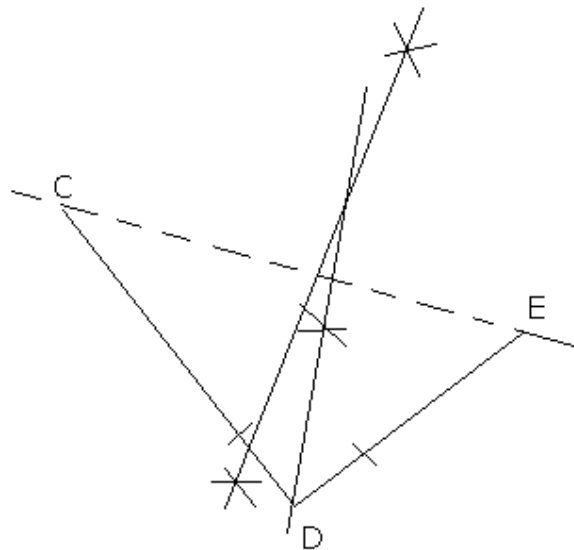
$$56 = 2 \times 3.14 \times r$$

$$r = \frac{56}{6.28}$$

$$r = \underline{8.92 \text{ cm}} \text{ correct to 2 decimal place}$$

[2]

12. [5]



13. For triangle ANV

$$\begin{aligned}
 n^2 &= v^2 + a^2 \\
 41^2 &= 9^2 + a^2 \\
 a^2 &= 41^2 - 9^2 \\
 a^2 &= (41 - 9)(41 + 9) \\
 a^2 &= 50 \times 32 \\
 a^2 &= 1600 \\
 a &= \sqrt{1600} \\
 a &= 40
 \end{aligned}$$

For triangle BNM

$$\begin{aligned}
 15^\circ &= 9^\circ + b^\circ \\
 b^\circ &= 225 - 81 \\
 b^\circ &= 144 \\
 b &= \sqrt{144} \\
 b &= 12
 \end{aligned}$$

Therefore, $MV = 40 - 12 = \underline{28m}$ Answer [3]

14. Mean = $\frac{\sum x}{n}$

$$\begin{aligned}
 8 &= \frac{10+8+7+12+x+6+10+4}{8} \\
 8 &= \frac{x+57}{8} \\
 x+57 &= 64 \\
 x &= 64 - 57 \\
 x &= \underline{7} \text{ Answer} \quad [2]
 \end{aligned}$$

15. (a) $a_n = a_1 + (n - 1)d$

$$\begin{aligned}
 &= 4 + (7 - 1) \times 3 \\
 &= 4 + 6 \times 3 \\
 &= 4 + 18 \\
 &= \underline{22} \text{ Answer} \quad [2]
 \end{aligned}$$

(b) $a_n = a_1 + (n - 1)d$

$$\begin{aligned}
 A_5 &= a_1 + (5 - 1) \times -2 \\
 20 &= a_1 + 4(-2) \\
 20 &= a_1 - 8 \\
 A_1 &= 20 + 8 \\
 A_1 &= 28
 \end{aligned}$$

Therefore, the terms are
 $28, (28 - 2 = 26), (26 - 2 = 24), (24 - 2 = 22)$
 $= \underline{28, 26, 24, 22}$ Answer [2]

Total = 50 marks



Answers to Paper 2

End of Grade 11 Mathematics Test Paper 2 Answers

1. (a) (b) $20.8\text{m} - 0.01355\text{m} = 20.78645\text{m}$
 $= \mathbf{20.786\text{m}}$ correct to 3 decimal places. [2]

(d) Speed of light = $300,000,000 \text{ m/s}$
 $= \frac{300,000,000 \times 3600}{1000}$
 $= 1,080,000,000 \text{ km/h}$
 Distance = time \times speed
 $= 2 \text{ hours} \times 1,080,000,000 \text{ km/h}$
 $= 216,000,000 \text{ km}$
 $= \mathbf{2.16 \times 10^8 \text{ km}}$ [2]

(d) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $a = 2$, $b = 5$, $c = -1$

$$x = \frac{-5 \pm \sqrt{5 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{-5 \pm \sqrt{25 + 8}}{4}$$

$$x = \frac{-5 \pm \sqrt{33}}{4}$$

$$x = \frac{-5 \pm 5.744}{4}$$

Either $x = \frac{-5 + 5.744}{4}$

$$x = \frac{0.744}{4}$$

$$= 0.186$$

$$= 0.19$$

Or $x = \frac{-5 - 5.744}{4}$

$$= \frac{-10.744}{4}$$

$$= -2.686$$

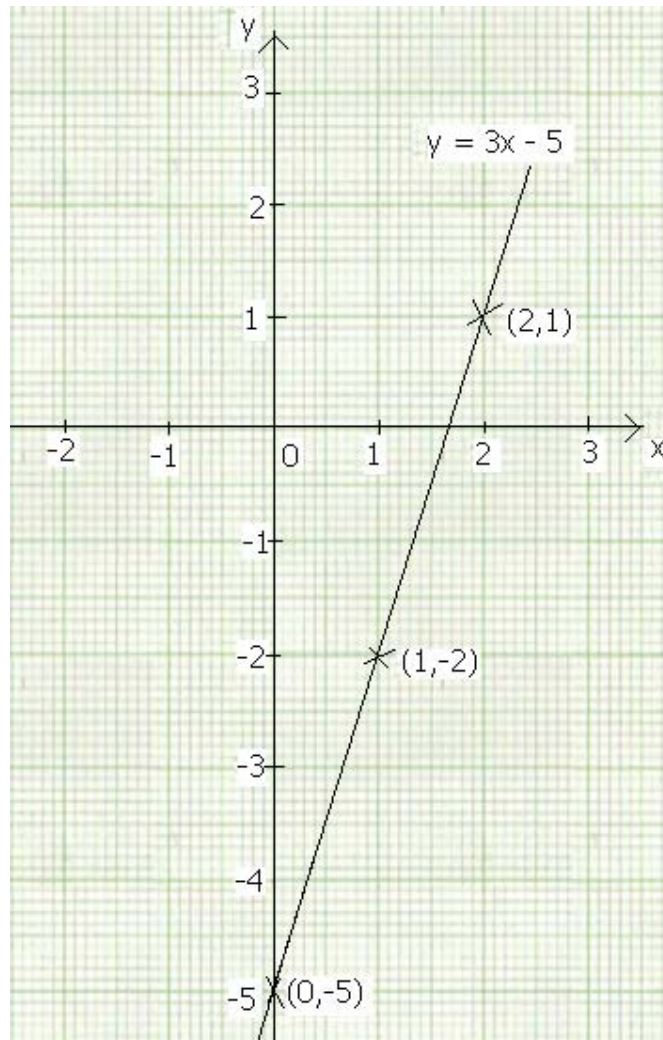
$$= -2.69$$

Therefore, $x = \mathbf{0.19 \text{ or } -2.69}$. [5]

2. (a) Table of values

X	0	1	2
Y	-5	-2	1

Scale: 1 cm represents 1 unit on the x-axis; 1 cm represents 2 units on the y-axis.



2(b) Using $c = \sqrt{a^2 + b^2}$ (Pythagoras Theorem)

$$\begin{aligned} \text{Then TR} &= \sqrt{1.5^2 + 4.2^2} \\ &= \sqrt{2.26 + 17.64} \\ &= \sqrt{19.89} \\ &= 4.459 \\ &= \mathbf{4.5 \text{ m}} \end{aligned}$$

[2]

(b) Using tangent ratio

$$\begin{aligned} \text{Tan}\langle \text{SQR} &= \frac{4.2}{(4.5+1.5)} \\ &= \frac{4.2}{6} \\ &= 0.7 \\ &= 34.99 \\ &= 35^\circ \end{aligned}$$

Therefore, $\langle \text{SQR} = 35^\circ$

[2]

(c) Using sine ratio

$$\begin{aligned} \text{Then } \sin 62^\circ &= \frac{6}{PQ} \\ PQ &= \frac{6}{\sin 62^\circ} \\ &= \frac{6}{0.8829} \\ &= 6.7957 \\ &= \mathbf{6.8m} \end{aligned}$$

[2]

3. (a) $\angle AOB = 2 \times \angle ADB = 2 \times 36^\circ = \mathbf{72^\circ}$ (angle at the centre of a circle is twice the angle on the circumference of the circle) [1]

(b) Since $\angle DAC = \angle DBC = 20^\circ$ (angle on the same segment of a circle are equal)

Then $\angle AED = 180^\circ - (36^\circ + 20^\circ) = 180^\circ - 56^\circ = \mathbf{124^\circ}$ (angle sum of a triangle) [1]

(c) Since $\angle FCO = 90^\circ$ (A tangent to a circle is perpendicular to the radius)

Then $\angle AFC = 180^\circ - \angle FCA + \angle ACF = 180^\circ - (20^\circ + 90^\circ) = 180^\circ - 110^\circ = \mathbf{70^\circ}$ [1]

(d) Since $\angle ADB = \angle ACB = 36^\circ$ (angles subtending on the same segment)

And $180^\circ - \angle AOB = 180^\circ - 72^\circ = 108^\circ$ (angle on a straight line)

Then $\angle OBD = 180^\circ - (36^\circ + 20^\circ + 108^\circ) = 180^\circ - 164^\circ = \mathbf{16^\circ}$ [1]

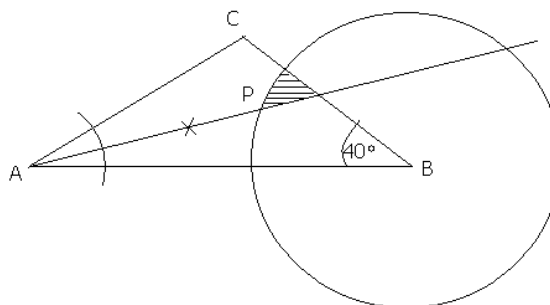
(e) Since $\angle OAB$ and $\angle OBA$ are base angle an isosceles triangle OAB,

Then $\angle OAB = \frac{1}{2}(180^\circ - 72^\circ) = \frac{1}{2} \times 108^\circ = \mathbf{54^\circ}$ [1]

(f) $\angle CAB = \angle TCB = 54^\circ$ (angle in an alternate segment)

Therefore $\angle BCT = \mathbf{54^\circ}$ [1]

4. (a) $AC = 5.8 \text{ cm}$ [2]



5. (a) $y = \frac{x^2}{5} + \frac{15}{x} - 8$

$$P = \frac{7^2}{5} + \frac{15}{7} - 8$$

$$P = 3.9$$

- (c) Draw tangent at (3,-12) and get any two points e.g.
from graph (2, -0.3), (3.8, -2)

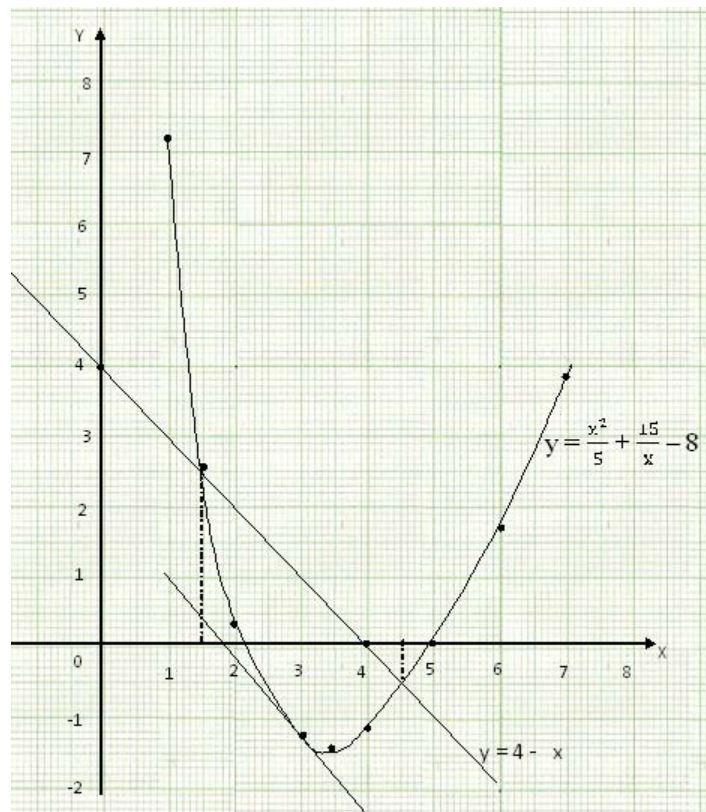
$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-0.3 - (-2)}{2 - 3.8} = \frac{-0.3 + 2}{2 - 3.8} \approx 0.94$$

- (d) Get solution where the line $y = 0$ (x-axis) meets)

$$y = \frac{x^2}{5} + \frac{15}{x} - 8. \quad x \approx 2.2, x \approx 5$$

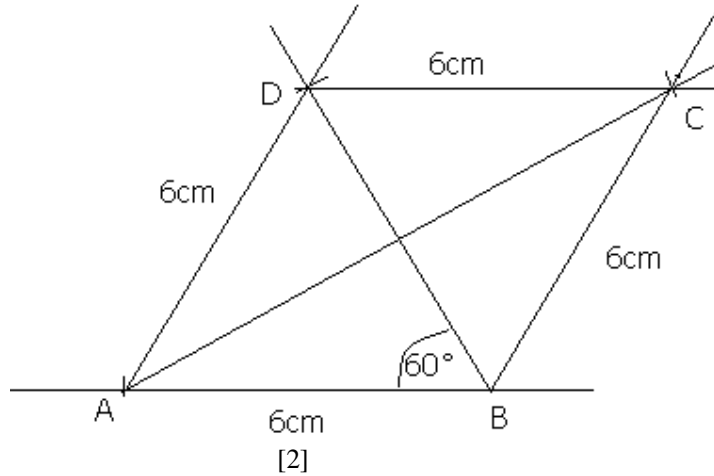
- (e) (ii) $x \approx 1.5, x \approx 4.6$

- (b) The graphs of $y = \frac{x^2}{5} + \frac{15}{x} - 8$ and $y = 4 - x$ are drawn as follows:



6. (a)

Scale: 1m: 2cm



(c) The quadrilateral is a **rhombus** [1]

(d) $AC = 10.5\text{cm}$ [2]

(e) Area = base \times height; where base = 3m, height = $3\text{m} \times \sin 60^\circ$

$$5 \quad \text{Area} = 3\text{m} \times (3\text{m} \times \sin 60^\circ)$$

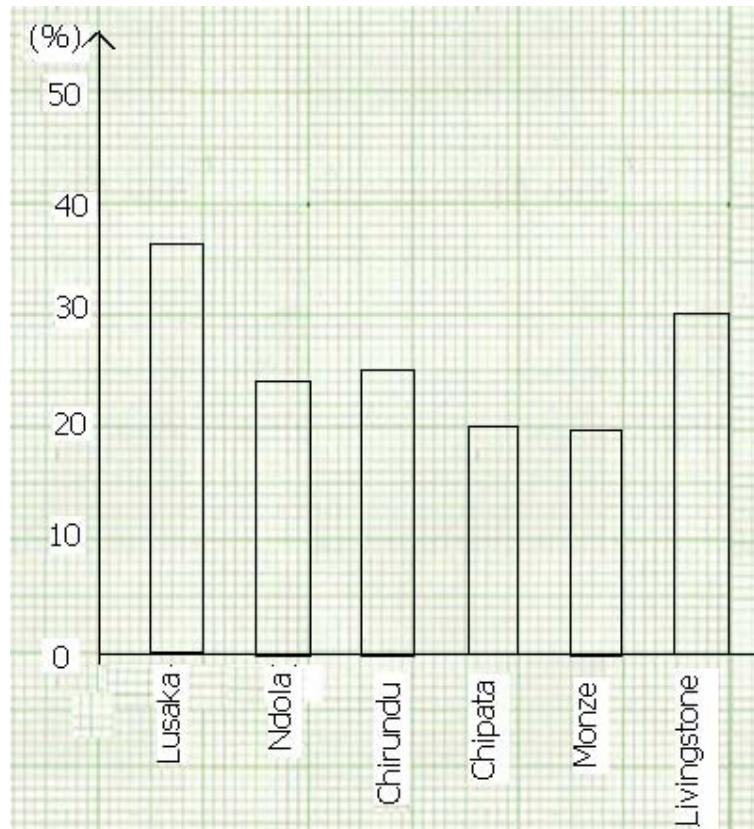
$$\text{Area} = 9 \times 0.866$$

$$\text{Area} = 7.79\text{m}^2$$

4 Therefore, the area of the floor is **7.79m^2**

[3]

7. (a)



(b) Total population = $8,000 \div 25\%$
 $= 8,000 \times \frac{100}{25}$
 $= 32,000$ [2]

(c) Monze = 20% of 30 000
 $= \frac{20}{100} \times 30\,000$
 $= 6\,000$

Livingstone = 25% of 65 000
 $= \frac{25}{100} \times 65\,000$
 $= 16\,250$

Therefore, total number of HIV/AIDS adults in Monze and Livingstone is $6\,000 + 16\,250 = \mathbf{22\,250}$ [3]

Mark	Frequency (f)	Mid-value (x)	(fx)
45-49	2	47	94

50-54	3	52	156
55-59	5	57	285
60-64	3	62	186
65-69	0	67	0
70-74	10	72	720
75-79	1	77	77
80-84	4	82	328
85-89	2	87	174
90-94	4	92	368
95-99	1	97	97
Totals	$\Sigma f = 35$	$\Sigma fx = 2\ 853$	

- (a) 35 learners [1]
 (b) Modal class mark is 70-74 [1]
 (c) Mean mark = $\frac{\Sigma fx}{\Sigma f} = \frac{2\ 853}{35} = 81.5$ [2]

8. (a) Outer surface area = $\pi r(r + 2h)$ where $r = 5$ cm, $h = 12$ cm

$$\begin{aligned} \text{Surface area} &= 3.142 \times 5(5 + 2 \times 12) \\ &= 15.71(5 + 24) \\ &= 15.71 \times 29 \\ &= \mathbf{455.59\ cm^3} \end{aligned} \quad [3]$$

- (b) Given $r = \frac{10-0.4}{2} = 4.8$ cm, $h = 12 - 0.2 = 11.8$ cm

$$\begin{aligned} V &= \pi r^2 h = 3.142 \times 4.8^2 \times 11.8 \\ &= 3.142 \times 23.04 \times 11.8 \\ &= 854.2218 \\ &= \mathbf{854.22\ cm^3} \text{ correct to 2 decimal} \\ &\quad \text{places} \end{aligned} \quad [3]$$

9. (a) The ratio 5cm to 1.5 km

$$\begin{aligned} 5\text{ cm} &: 1.5\text{ km} \times 100\ 000 \text{ (Since } 1\text{ km} = 100\ 000\text{ cm)} \\ 5\text{ cm} &: 150\ 000\text{ cm} \\ &\mathbf{1: 30\ 000} \end{aligned} \quad [2]$$

- (b) 12 men take 10 days

1 man takes $12 \times 10 = 120$ days

15 men takes = $120 \div 15 = \mathbf{8 \text{ days}}$ [2]

(c) $v \propto \frac{x^2}{y}$
 $v = \frac{x^2 k}{y}$
 $9 = \frac{3^2 \times k}{4}$
 $K = 4$
 Therefore, $v = \frac{4x^2}{y}$
 $50 = \frac{4x^2}{2}$
 $x^2 = \frac{50 \times 2}{4}$
 $x = \sqrt{25}$
 $x = \mathbf{5}$ [4]

10. (a) The ratio of their areas are 81:49 [1]

(b) (i) $\angle SPQ = \angle SQR$ or $\angle PRS = \angle PQR$ [1]

(ii) $12 : 18 = PS : 12$
 $\frac{12}{18} = \frac{PS}{12}$
 $PS = \frac{12 \times 12}{18}$
 $PS = \mathbf{8 \text{ cm}}$ [3]

(iii) Their ratio is 12: 18 = 2:3

The ratio of their areas is $2^2 : 3^2 = 20 : x$

$$4 : 9 = 20 : x$$

$$x = \frac{20 \times 9}{4} = 45$$

Therefore, the area of triangle PQR = $\mathbf{45 \text{ cm}^2}$

[3]

11. (a) $a_n = a + (n - 1)d$; where $a = 2$, $n = 20$ and $d = 3$

$$\begin{aligned} a_{20} &= 2 + (20 - 1) \times 3 \\ &= 2 + 19 \times 3 \\ &= 2 + 57 \\ &= \mathbf{114} \end{aligned}$$
 [2]

(b) 5, 9, 13, 17, 21, 25, ...

$a_n = a + (n - 1)d$; where $a = 5$, $d = 4$

$$a_n = 5 + (n - 1)4$$

$$a_n = 5 + 4n - 4$$

$$\mathbf{a_n = 4n + 1}$$
 [2]

Total = 100 marks